Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimization Theory
Spring 2013

Exercise set 7

Tutorial: Thursday 07.03 16:15-17:00 in Kjl 4.

1 Problem 12.17 in N&W p. 353.

Solution: From Eqn. 12.31 we have that λ^* *satisfies the system*

$$\sum_{i \in \mathcal{A}(x^*)} \lambda_i^* \nabla c_i(x^*) = \nabla f(x^*).$$

The solution is unique when $\{\nabla c_i(x^*)\}_{i \in \mathcal{A}(x^*)}$ *are linearly independent (which is the LICQ condition!). Then the equation*

$$\sum_{i \in \mathcal{A}(x^*)} z_i \nabla c_i(x^*) = 0$$

will only have the 0*-solution, and* λ^* *is unique.*

2 (Midterm Exam 2010)

Consider the following constrained optimization problem for $(x_1, x_2) \in \mathbb{R}^2$:

$$\min_{x \in \Omega} \{-4x_1 - x_2\}, \tag{1}$$

where Ω is defined in terms of the constraints

$$0 \le x_1 \le 2, \tag{2}$$
$$0 \le x_2, \tag{3}$$
$$x_2 \le 3 - x_1. \tag{4}$$

a) Reformulate the constraints into four constraints of the form

$$c_i(x) \ge 0, \; i = 1, \cdots, 4, \tag{5}$$

and write down all KKT-equations and inequalities.

b) Solve the problem graphically by making a sketch of Ω.

c) Identify the active and inactive constraints and the corresponding Lagrange multipliers at the solution.

Solution: ***(a)*** *The constraints may be written*

$$\begin{aligned} c_1(x) &= x_1 \geq 0, && (6)\\ c_2(x) &= 2 - x_1 \geq 0, && (7)\\ c_3(x) &= x_2 \geq 0, && (8)\\ c_4(x) &= 3 - x_1 - x_2 \geq 0. && (9) \end{aligned}$$

Hence, the Lagrangian is

$$\mathcal{L}(x, \lambda) = -4x_1 - x_2 - \lambda_1 x_1 - \lambda_2 (2 - x_1) - \lambda_3 x_2 - \lambda_4 (3 - x_1 - x_2), \tag{10}$$

and $\nabla_x \mathcal{L}(x, \lambda) = 0$ *gives the equations*

$$\begin{aligned} -\lambda_1 + \lambda_2 + \lambda_4 &= 4, && (11)\\ -\lambda_3 + \lambda_4 &= 1, && (12) \end{aligned}$$

along with the rest of the KKT equations:

$$\begin{aligned} \lambda_1 x_1 &= 0, && (13)\\ \lambda_2 (2 - x_1) &= 0, && (14)\\ \lambda_3 x_2 &= 0, && (15)\\ \lambda_4 (3 - x_1 - x_2) &= 0, && (16) \end{aligned}$$

plus all 4 inequalities in Eqn. (6)–(9), and the requirements $\lambda_1, \cdots, \lambda_4 \geq 0$.

(b) *The function* $f(x)$ *has level curves defined by*

$$-4x_1 - x_2 = const., \tag{17}$$

and the negative gradient direction is therefore constant,

$$-\nabla f' = 4\mathbf{i} + \mathbf{j}. \tag{18}$$

This, along with the constraints in Eqn. (6)–(9) that defines Ω *is shown in Fig. 1. The solution is clearly* $x^* = (2, 1)'$ *with* $f(x^*) = -4 \times 2 - 1 = -9$.

(c) *Eqns. (13)–(16) give that* $\lambda_1 = \lambda_3 = 0$ *(*c_1 *and* c_3 *are not active), whereas* c_2 *and* c_4 *are active, so that* λ_2 *and* λ_4 *may be different from* 0. *It then follows from Eqns. (11) and (12) that*

$$\lambda_4 = 1 \text{ and } \lambda_2 = 3.$$

As a final check,

$$\begin{aligned} \nabla f(x^*)' &= \lambda_2 \nabla c_2 (x^*)' + \lambda_4 \nabla c_4 (x^*)' \\ &= 3(-\mathbf{i}) + 1 \times (-\mathbf{i} - \mathbf{j}) = -4\mathbf{i} - \mathbf{j}. && (19) \end{aligned}$$

3 Problem 12.21 in N&W, p. 354.

Solution: *First change the sign of the objective function* $g(x) = -f(x) = -x_1 x_2$.

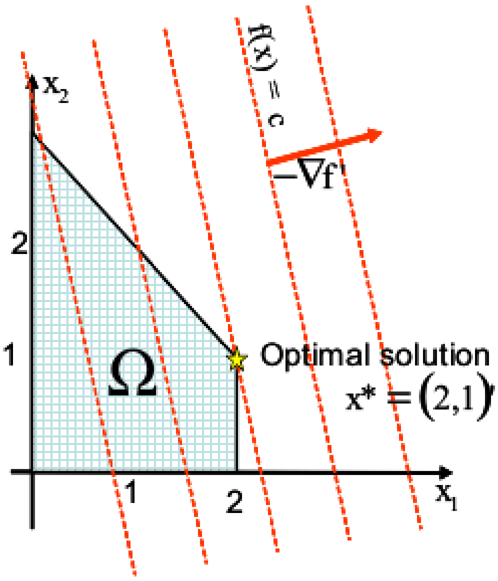

Figure 1: Graph of the level curves of f, the constant negative gradient vector $-\nabla f'$, and Ω.

We easily see that $(0,0)$ *is the only interior KKT-point, but this is a* saddle-point. *On the boundary of* Ω *the KKT-equations are*

$$\begin{aligned}\nabla_x\left[-x_1x_2-\lambda\left(1-x_1^2-x_2^2\right)\right]&=0,\\ \lambda\left(1-x_1^2-x_2^2\right)&=0,\\ \lambda&\geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}-x_2+2\lambda x_1&=0,\\ -x_1+2\lambda x_2&=0,\\ x_1^2+x_2^2&=1,\\ \lambda&>0.\end{aligned}$$

The solutions are $(1/\sqrt{2},1/\sqrt{2})$, $\lambda=1/2$ *and* $(-1/\sqrt{2},-1/\sqrt{2})$, $\lambda=1/2$. *It is quite obvious that these are minima for* g *and maxima for* f. *In order to be complete, one should also check the tangent directions to the circle using the second order conditions. Alternatively we could, along the circle, introduce* $x_1=\cos\theta$ *and* $x_2=\sin\theta$, *and observe that* $f(x_1,x_2)$ *is simply equal to* $\frac{1}{2}\sin(2\theta)$.

4 Consider the problem

$$\begin{aligned}&\min(x_2+x_3),\\ x\in\Omega=&\left\{x\;;\;x_1+x_2+x_3=1,\;x_1^2+x_2^2+x_3^2\geq 1\right\}.\end{aligned}$$

Note that the feasible domain is unbounded.

a) Show that the only KKT-point for the problem is $(-1,2,2)^{\mathrm{T}}/3$.

b) Use the second order conditions to investigate whether this KKT-point really is a local minimum.

Solution: Note that Ω is unbounded, and that $f(x)$ is unbounded below on Ω. We can only hope for local *minima, and we observe that Eqn. 12.30a will be*

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \lambda_1 + \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} \lambda_2.$$

First of all, there are no solutions for $\lambda_2 = 0$. Assuming $\lambda_2 \neq 0$, we see from the 2nd and 3rd equations above that $x_2 = x_3$. Moreover, the inequality constraint is active since $\lambda_2 > 0$. Hence, writing $x = (y, z, z)'$ we have

$$\begin{aligned} \lambda_1 + 2y\lambda_2 &= 0, \\ \lambda_1 + 2z\lambda_2 &= 1, \\ y + 2z &= 1, \\ y^2 + 2z^2 &= 1. \end{aligned}$$

The two last equations give the solutions

$$x_a = (1,0,0)', \; x_b = (-1,2,2)/3.$$

The multipliers are solved from the first pair of equations,

$$\lambda_a = (1, -1/2), \; \lambda_b = (1/3, 1/2).$$

The only KKT-point is therefore x_b, but is it a local minimum?

We check the Hessian of the Lagrange function

$$\begin{aligned} &\nabla_x^2 \left[x_2 + x_3 - \lambda_1^* (x_1 + x_2 + x_3 - 1) - \lambda_2^* \left(x_1^2 + x_2^2 + x_3^2 - 1 \right) \right] \\ &= -2\lambda_2^* I_{3\times 3} = -I_{3\times 3}. \end{aligned}$$

The gradients of the constraints in x_b are

$$\begin{aligned} \nabla c_1 (x_b) &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \\ \nabla c_1 (x_b) &= \frac{2}{3} \begin{pmatrix} -1 & 2 & 2 \end{pmatrix}, \end{aligned}$$

and the matrix A (see the notes) is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{2}{3} & \frac{4}{3} & \frac{4}{3} \end{bmatrix}.$$

Since A has rank 2, the null space $\mathcal{N}(A)$ is spanned by $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}'$. But no (non-zero) projection of $-I_{3\times 3}$ will ever be positive semi-definite, so we have to conclude that no local minimum exists.