Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimization Theory
Spring 2013

Exercise set 8

Tutorial: Thursday 14 16:15-17:00 in Kjl 4.

$\boxed{1}$ Let $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $\|x\|^2 = \sum_{i=1}^n x_i^2$, $\{y_k\}_{k=1}^K$ be a set of K given vectors in $\mathbb{R}^n$, and define $f(x)$ by

$$f(x) = \frac{1}{2K} \sum_{k=1}^{K} \|x - y_k\|^2.$$

Consider the problem

$$\min f(x)$$

when

$$Ax = b,$$

and A is an $r \times n$ matrix with (full) rank $r < n$.

a) Show, without using part **b)**, that this problem has a unique solution.
Hint: Show that the feasible domain $\Omega = \{x \mid Ax = b\}$ is *convex and non-empty*, and that the function f is *strictly convex* and tends to infinity when $\|x\| \to \infty$.

Solution: First note that

$$\nabla f(x) = \frac{1}{K} \sum_{k=1}^{K} (x - y_k) = x - \bar{x},$$

$$\bar{x} = \frac{1}{K} \sum_{k=1}^{K} y_k.$$

Hence, $\nabla^2 f(x) = I$*, and* f *is* strictly convex.

Moreover, $\Omega = \{x \mid Ax = b\} \neq \varnothing$*, since* A *has full rank. It is easy to see that* Ω *is convex since* $\alpha x_1 + (1-\alpha)x_2 \in \Omega$ *if* $x_1, x_2 \in \Omega$ *and* $0 \leq \alpha \leq 1$*. Thus,* Ω *is convex and the problem has a unique solution, since* $f(x) \to \infty$ *when* $\|x\| \to \infty$.

b) Write down the KKT conditions for the problem and show that the unique solution x^* is

$$x^* = \bar{x} + A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}(b - A\bar{x}),$$

$$\bar{x} = \frac{1}{K} \sum_{k=1}^{K} y_k.$$

Solution: *We have from the above that* $\nabla f(x) = x - \bar{x}$. *The equality constraints are*

$$c(x) = Ax - b = 0.$$

We have

$$\nabla c = \begin{bmatrix} (\nabla c_1)^{\mathrm{T}} \\ (\nabla c_2)^{\mathrm{T}} \\ \vdots \\ (\nabla c_r)^{\mathrm{T}} \end{bmatrix} = A.$$

The Lagrangian function is

$$\mathcal{L}(x, \lambda) = f(x) - \lambda^{\mathrm{T}} c(x),$$

and the KKT conditions become

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= x^* - \bar{x} - A^{\mathrm{T}}\lambda^* = 0, \\ Ax^* &= b. \end{aligned}$$

From this, we solve for λ^* *by left-multiplying by* A *and using that* AA^{T} *is non-singular:*

$$b - A\bar{x} - (AA^{\mathrm{T}})\lambda^* = 0 \quad \Longrightarrow \quad \lambda^* = (AA^{\mathrm{T}})^{-1}(b - A\bar{x}).$$

If this is inserted into the first KKT condition, we obtain

$$x^* = \bar{x} + A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}(b - A\bar{x}).$$

c) How will the KKT conditions be modified if some of the equations are replaced by inequalities, e.g.

$$a_i x - b_i \geq 0, \quad i \in \mathcal{I} \subset \{1, \ldots, r\},$$

where a_i is row vector i of A? When will x^* still be a solution?

Solution: *If some of the equations are replaced by inequalities, the KKT conditions become*

$$\begin{aligned} x^* - \bar{x} - A^{\mathrm{T}}\lambda^* &= 0, \\ a_i x^* - b_i &= 0, \quad i \in \mathcal{E}, \\ a_i x^* - b_i &\geq 0, \quad i \in \mathcal{I}, \\ \lambda_i^* &\geq 0, \quad i \in \mathcal{I}, \\ \lambda_i^*(a_i x^* - b_i) &= 0, \quad i = 1, \ldots, r. \end{aligned}$$

Then x^* *will continue to be a solution if the* λ_i^* *corresponding to the inequality constraints are zero. The domain is convex and the function is still strictly convex, and any solution will be unique. Clearly, the objective function is unbounded when* $\|x\| \to \infty$, *so that it certainly has a minimum somewhere. (As shown in the notes, for convex problems, the KKT conditions are also sufficient for a KKT point to be the solution.)*

$\boxed{2}$ **a)** Write the LP problem

$$\begin{aligned} &\max(-x_1 + x_2), \\ &x_1 \leq 2 + x_2, \\ &6 - x_2 \geq x_1, \end{aligned}$$

on standard form,

$$\begin{aligned} &\min(c^{\mathrm{T}} x), \\ &Ax = b, \\ &x \geq 0. \end{aligned}$$

Solution: *We change* max *to* min, *write* x_1 *and* x_2 *as differences between positive numbers, and introduce slack variables in the inequalities:*

$$\begin{aligned} &\min\big((y_1 - y_2) - (y_3 - y_4)\big), \\ &(y_1 - y_2) - (y_3 - y_4) + y_5 = 2, \\ &(y_1 - y_2) + (y_3 - y_4) + y_6 = 6. \end{aligned}$$

Then

$$\begin{aligned} &\min c^{\mathrm{T}} y, \\ &Ay = b, \\ &y \geq 0, \end{aligned}$$

where

$$c^{\mathrm{T}} = (1, -1, -1, 1, 0, 0),$$
$$A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 \end{bmatrix}, \quad b = (2, 6)^{\mathrm{T}}.$$

b) Solve Problem 13.1 (p. 389) in N&W.

Solution: *We first try to get rid of the double bound on* y *and observe that we may write*

$$\begin{aligned} &y = l + z, \\ &z + s = u - l, \\ &z \geq 0, \\ &s \geq 0. \end{aligned}$$

In addition, we have to introduce $x = x_1 - x_2$, *and a slack variable in the inequality constraint,*

$$A_2 x + B_2 y + r = b_2, \quad r \geq 0.$$

The vector in the objective function takes the form

$$\tilde{c}^{\mathrm{T}} = (-c^{\mathrm{T}}, c^{\mathrm{T}}, -d^{\mathrm{T}}, 0^{\mathrm{T}}, 0^{\mathrm{T}}).$$

(A constant term $-d^{\mathrm{T}} l$ *is of no interest.) The system of equations is then*

$$\begin{bmatrix} A_1 & -A_1 & 0 & 0 & 0 \\ A_2 & -A_2 & B_2 & I & 0 \\ 0 & 0 & I & 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \\ r \\ s \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - B_2 l \\ u - l \end{bmatrix},$$

where $x_1 \geq 0$, $x_2 \geq 0$, $z \geq 0$, $r \geq 0$, *and* $s \geq 0$.

$\boxed{3}$ Determine and solve the *dual* problem to the *primal* problem

$$\begin{aligned} &\min(5x_1 + 3x_2 + 4x_3), \\ &x_1 + x_2 + x_3 = 1, \\ &x_i \geq 0, \quad i = 1, 2, 3. \end{aligned}$$

Solution: The problem is already in standard form

$$\begin{aligned} \min c^{\mathrm{T}} x&, \\ Ax = b&, \\ x \geq 0&, \end{aligned}$$

where

$$\begin{aligned} c^{\mathrm{T}} &= (5, 3, 4), \\ A &= (1, 1, 1), \\ b &= 1. \end{aligned}$$

The dual problem is therefore

$$\begin{aligned} \max b^{\mathrm{T}} \lambda &= \max \lambda, \\ A^{\mathrm{T}} \lambda &\leq c. \end{aligned}$$

Since the inequalities give $\lambda \leq 3$, *the solution is simply* $\lambda = 3$. *Note that*

$$(c - A^{\mathrm{T}} \lambda)_i x_i = 0, \quad i = 1, 2, 3,$$

that is, $x^* = (0, x_2^*, 0)^{\mathrm{T}}$, *and since* $Ax = b$, $x_2^* = 1$.

$\boxed{4}$ Solve the following LP problem by using the MATLAB Optimization Toolbox:

$$\begin{aligned} \min(c^{\mathrm{T}} x)& \\ Ax \leq b&, \\ x \geq 0&, \end{aligned}$$

where

$$c^{\mathrm{T}} = \begin{bmatrix} -120 & 32 & -48 & -64 \end{bmatrix},$$

$$A = \begin{bmatrix} 3 & 2 & -1 & 4 \\ -1 & 2 & 3 & 12 \\ 4 & 3 & 5 & 21 \\ 8 & 3 & 4 & 5 \\ 6 & 2 & 4 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 42 \\ 36 \\ 45 \\ 28 \\ 14 \end{bmatrix}.$$

Solution: In the MATLAB *Optimization Toolbox,* `linprog` *may be used as follows:*

```
[x,fval,exitflag,output,lambda] = linprog(c,A,b,[],[],zeros(size(c)),[]);
```

This produces

```
x =
    2.0410
    0.0000
    0.0000
    1.7541

fval =
 -357.1803

exitflag =
     1

output =
         iterations: 8
          algorithm: 'large-scale: interior point'
       cgiterations: 0
            message: 'Optimization terminated.'
    constrviolation: 0
     firstorderopt: 1.0568e-08
```

and the structure `lambda`*. The algorithm used is a simplex algorithm. The values of the Lagrange multipliers are obtained in* `lambda.ineqlin` *and* `lambda.lower`*:*

```
lambda.ineqlin =
    0.0000
    0.0000
    2.1639
    0.0000
   18.5574

lambda.lower =
    0.0000
   75.6066
   37.0492
    0.0000
```

This is in accordance with the KKT conditions.