Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4180 Optimization Theory
Spring 2013

Exercise set 10

Tutorial: Thursday 25 16:15-17:00 in El 1 (NB! Change of room).

1 *Troutman*, Problem 1.2.

Hints: The transit time from $(0,0)$ to $(1,1)$ along a path $y(x)$, where $y(0) = 0$, $y(1) = 1$, is given by

$$T = \frac{1}{\sqrt{2g}} \int_0^1 \left(\frac{1 + y'(x)^2}{y(x)} \right)^{1/2} \mathrm{d}x,$$

and the problem is a technical exercise in estimating the value of this integral for various paths, $y = y(t)$.

For (c) you may use that

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{(\sin\theta)^{1/2}} = \frac{1}{2}\pi^{3/2}\frac{\sqrt{2}}{\Gamma(3/4)^2} \approx 2.622.$$

Point (e) seems to be tricky, so try the not-so-obvious inequality

$$\sin\theta \geq \theta - \theta^2/\pi, \quad 0 \leq \theta \leq \pi/2.$$

Maybe you see a simpler way!

Solution: The transit time from $(0,0)$ *to* $(1,1)$ *along a path* $y(x)$*, where* $y(0) = 0$, $y(1) = 1$*, is given by*

$$T = \frac{1}{\sqrt{2g}} \int_0^1 \left(\frac{1 + y'(x)^2}{y(x)} \right)^{1/2} \mathrm{d}x.$$

a) *For the straight line path* $y = x$*, we thus obtain*

$$T_{sl} = \frac{1}{\sqrt{2g}} \int_0^1 \left(\frac{1+1}{x} \right)^{1/2} \mathrm{d}x = \frac{2}{\sqrt{g}}.$$

b) and c) *For a circular quarter-arc,*

$$x = 1 - \cos\theta, \quad y = \sin\theta,$$

we obtain

$$\begin{aligned} T_{ca} &= \int_{(0,0)}^{(1,1)} \frac{\mathrm{d}s}{v(s)} = \int_{(0,0)}^{(1,1)} \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{\sqrt{2gy}} \\ &= \frac{1}{\sqrt{2g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\sin\theta}} \\ &= \frac{1}{\sqrt{2g}} \left(\frac{1}{2}\pi^{3/2} \frac{\sqrt{2}}{\Gamma(3/4)^2} \right) \approx \frac{2}{\sqrt{g}} 0.927. \end{aligned}$$

(The integral came out from Maple, and is only checked versus a direct numerical integration. It is a special case of an elliptic integral.)

d) *The inequality*

$$\sin\theta < \theta$$

holds for all $\theta \in (0, \pi/2)$, and then

$$\frac{1}{\sqrt{2g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\theta}} < \frac{1}{\sqrt{2g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\sin\theta}} = T_{ca}.$$

Now, $\int_0^{\pi/2} \theta^{-1/2}\,\mathrm{d}\theta = \sqrt{2\pi}$, and we obtain

$$\frac{1}{\sqrt{2g}} \sqrt{2\pi} \approx \frac{2}{\sqrt{g}} \cdot 0.886 < T_{ca}.$$

e) *One way to attack an opposite inequality would be to find a function $h(\theta)$, $h(\theta) \leq \sin\theta$, such that*

$$T_{ca} = \frac{1}{\sqrt{2g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\sin\theta}} \leq \frac{1}{\sqrt{2g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{h(\theta)}} \leq \frac{2}{\sqrt{g}}.$$

A first try is the well-known

$$\sin\theta \geq \frac{2}{\pi}\theta.$$

But,

$$\frac{1}{\sqrt{2g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{2\theta/\pi}} = \frac{1}{\sqrt{2g}}\pi = \frac{2}{\sqrt{g}} \frac{\pi}{2\sqrt{2}} \approx \frac{2}{\sqrt{g}} \cdot 1.11,$$

so this is too crude.

Another idea could be to introduce the substitution $u = \sin\theta$:

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\sin\theta}} = \int_0^1 \frac{1}{\sqrt{u}} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u,$$

This does not seem much simpler, since $(1-u^2)^{-1/2}$ goes from 1 to ∞ when u increases from 0 to 1.

The next try is the more accurate inequality (prove it!)

$$\sin\theta \geq \theta - \frac{\theta^3}{6}, \quad 0 \leq \theta \leq \pi/2,$$

and this will in fact work, but the integral

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\theta - \theta^3/6}}.$$

is not elementary.

It would be easier if the power of θ is 2 instead of 3, so we consider still another inequality,

$$\sin\theta \geq \theta - \frac{\theta^2}{\pi}, \quad 0 \leq \theta \leq \pi/2$$

(Prove this by taking the derivative of each side.) This one works, and the integral can be computed analytically:

$$\frac{1}{\sqrt{2g}} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{\theta - \theta^2/\pi}} = \frac{1}{\sqrt{2g}} \frac{1}{2} \pi^{3/2} \approx \frac{2}{\sqrt{g}} \cdot 0.984.$$

There are probably simpler ways to see this!

2 *Troutman*, Problem 2.5 (a), (c), (e).

Hints: In some of these and following problems you'll need to put $\mathrm{d}/\mathrm{d}\varepsilon$ inside the integral sign,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_a^b h(x,\varepsilon)\,\mathrm{d}x = \int_a^b \frac{\partial h(x,\varepsilon)}{\partial\varepsilon}\,\mathrm{d}x. \tag{1}$$

Theorem A.13 in Troutman is a simple sufficient condition for this to be allowed: *Assume that $[a,b]$ is finite and h as well as $\partial h/\partial\varepsilon$ are continuous on $[a,b]\times[\alpha,\beta]$. Then (1) holds for all $\varepsilon \in [\alpha,\beta]$.*

All problems are most easily solved by applying the formula

$$\delta J(y;v) = \frac{\partial}{\partial\varepsilon} J(y+\varepsilon v)\bigg|_{\varepsilon=0}.$$

Solution:

a)

$$\begin{aligned}\delta J(y;v) &= \frac{\partial}{\partial\varepsilon}\big(y(a)+\varepsilon v(a)\big)^3\bigg|_{\varepsilon=0} \\ &= 3v(a)\big(y(a)+\varepsilon v(a)\big)^2\bigg|_{\varepsilon=0} \\ &= 3y(a)^2 v(a)\end{aligned}$$

c) *We assume that $[a,b]$ is a finite interval, and we compute the derivative of the integrand,*

$$\frac{\partial}{\partial\varepsilon}\sqrt{2+x^2-\sin\big(y'(x)+\varepsilon v'(x)\big)} = \frac{-\cos\big(y'(x)+\varepsilon v'(x)\big)}{2\sqrt{2+x^2-\sin\big(y'(x)+\varepsilon v'(x)\big)}} v'(x).$$

This will always be a continuous, and hence bounded, function since

$$\left|\frac{-\cos\big(y'(x)+\varepsilon v'(x)\big)}{2\sqrt{2+x^2-\sin\big(y'(x)+\varepsilon v'(x)\big)}} v'(x)\right| \leq \tfrac{1}{2}|v'(x)| \in C(a,b).$$

We may therefore apply the note about differentiation and find immediately

$$\delta J(y;v) = \int_a^b \frac{-\cos\big(y'(x)\big)}{2\sqrt{2+x^2-\sin\big(y'(x)\big)}} v'(x)\,\mathrm{d}x$$

e) *A similar argument as in* **c)** *also applies here*

$$\begin{aligned}\delta J(y;v) &= \frac{\partial}{\partial\varepsilon}\int_a^b \left(x^2\big(y(x)+\varepsilon v(x)\big)^2 + \mathrm{e}^{y'(x)+\varepsilon v'(x)}\right)\mathrm{d}x\bigg|_{\varepsilon=0} \\ &= \int_a^b \frac{\partial}{\partial\varepsilon}\left(x^2\big(y(x)+\varepsilon v(x)\big)^2 + \mathrm{e}^{y'(x)+\varepsilon v'(x)}\right)\bigg|_{\varepsilon=0}\mathrm{d}x \\ &= \int_a^b \big(2x^2 y(x)v(x) + \mathrm{e}^{y'(x)}v'(x)\big)\,\mathrm{d}x.\end{aligned}$$

3 *Troutman*, Problem 2.10 (a).
Hint: Use that $J(y+\varepsilon v) - J(y) = \varepsilon\delta J(y;v) + o(\varepsilon)$.

Solution: We recognize the well-known differentiation rules also for the Gateaux derivative. In all proofs of this sort, the trick is to start from the definition. *Since we know that*

$$\lim_{\varepsilon\to 0}\frac{J(y+\varepsilon v)-J(y)}{\varepsilon} = \delta J(y;v),$$

we also must have $J(y+\varepsilon v) - J(y) = \varepsilon\delta J(y;v) + o(\varepsilon)$. *This is exactly what we need:*

$$\begin{aligned}\delta(J\tilde{J})(y;v) &= \lim_{\varepsilon\to 0}\frac{J(y+\varepsilon v)\tilde{J}(y+\varepsilon v) - J(y)\tilde{J}(y)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{\big(J(y)+\varepsilon\delta J(y;v)+o(\varepsilon)\big)\big(\tilde{J}(y)+\varepsilon\delta\tilde{J}(y;v)+o(\varepsilon)\big) - J(y)\tilde{J}(y)}{\varepsilon} \\ &= \delta J(y;v)\tilde{J}(y) + J(y)\delta\tilde{J}(y;v) + \lim_{\varepsilon\to 0}\frac{o(\varepsilon)}{\varepsilon} \\ &= \delta J(y;v)\tilde{J}(y) + J(y)\delta\tilde{J}(y;v).\end{aligned}$$

With this as an example, points **b)** *and* **c)** *in the same problem are easy.*

4 *Troutman*, Problem 2.12.
Hint: Consider also the convexity of this functional.
Solution:

a) *Assume that* y_0 *is a solution and consider*

$$\begin{aligned}F(y_0+v) - F(y_0) &= \int_0^b \big(\rho(y_0+v)^2 + \beta(y_0+v) - \rho y_0^2 - \beta y_0\big)\,\mathrm{d}x \\ &= \int_0^b \big((2\rho y_0+\beta)v + \rho v^2\big)\,\mathrm{d}x.\end{aligned}$$

We see that y_0 *is a minimum if*

$$2\rho y_0 + \beta = 0,$$

that is,

$$y_0(x) = -\frac{\beta(x)}{2\rho(x)}.$$

b) *This function is feasible since $\rho, \beta \in C^1[a,b]$ and a unique minimum, since $\rho(x) > 0$, and*

$$\int_0^b \rho v^2 \, \mathrm{d}x = 0$$

if and only if $v(x) = 0$ for all $x \in [a,b]$.

c) *If $\rho < 0$, then F is strictly concave and y_0 is a global maximum.*

5 *Troutman*, Problem 3.6.

Solution: *The integrand $f(x,y,z) = z^2/x$ is strongly convex for $x \in [1,2]$ since*

$$f(x, y+v, z+w) - f(x,y,z) = \frac{\partial f}{\partial z} w + \frac{w^2}{x} \geq \frac{\partial f}{\partial z} w,$$

and equality occurs only for $w = 0$. Thus, F is strictly convex for the standard case with two fixed end points. The Euler–Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}x} f_{y'} - f_y = 2 \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y'}{x} \right) = 0.$$

With $x \in [1,2]$, this immediately gives

$$\frac{y'}{x} = A,$$

and

$$y(x) = \frac{A}{2} x^2 + B.$$

a) *For fixed endpoints, $y(1) = 0$, $y(2) = 3$ and the solution becomes*

$$y(x) = x^2 - 1.$$

b) *Here $y(2) = 3$, whereas $y(1)$ is free, and we need the natural boundary condition*

$$f_{y'}\big(1, y(1), y'(1)\big) = \frac{2y'(1)}{1} = 0.$$

Since $y'(1) = 0$, $y(x) = 0x^2 + B$, and hence

$$y(x) = 3.$$

*Both solutions in **a)** and **b)** are unique.*

c) *When both endpoints are free, $y'(1) = y'(2) = 0$ is necessary to ensure that $\delta F(y; v) = 0$. Interestingly enough, any function*

$$y(x) = B$$

will now be a solution. Thus, when both endpoints are free, partial strong convexity for f is not sufficient for F to be strictly convex on the set of feasible functions since this set may become too big.

6 *Troutman*, Problem 3.7.

Solution: For fixed x

$$f(x,y,z) = 2\mathrm{e}^x y + z^2$$

is the sum of a linear (in y) and a partially strongly convex function. This implies that F is strictly convex, at least for situation $\mathcal{D}$ and $\mathcal{D}_1$. The Euler–Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}x} f_{y'} - f_y = \frac{\mathrm{d}}{\mathrm{d}x}(2y') - 2\mathrm{e}^x = 0,$$

or

$$y'' = \mathrm{e}^x.$$

The general solution is

$$y(x) = \mathrm{e}^x + Ax + B.$$

a) *Here $y(0) = 0$ and $y(1) = 1$, and the (unique) solution becomes*

$$y(x) = \mathrm{e}^x - 1 + (2 - \mathrm{e})x.$$

b) *Since $y(1)$ is free, we impose the natural condition $2y'(1) = 0$, and with $y(0) = 0$, the solution is*

$$y(x) = \mathrm{e}^x - \mathrm{e}x - 1.$$

c) *(This point is not included in the problem but in the solution at the end of the book.) If we consider this problem with both endpoints free, we need $y'(0) = y'(1) = 0$. However, this can* not *be achieved for any set of constants in the general solution of the Euler–Lagrange equation, and no solution exists.*

7 *Troutman*, Problem 3.28.

Hint: Verify that the given solution satisfies the Euler equation and the constraints.

Solution: The function $f(y,z) = \sqrt{1+z^2}$ is strongly convex, and so is also $\sqrt{1+z^2} + \lambda y$ since the last term is linear. Thus, $F + \lambda G$ will be strictly convex (for fixed endpoints) and any unique solution of the Euler–Lagrange equation a global minimum. The equation $\delta F + \lambda \delta G = 0$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) - \lambda = 0,$$

that is,

$$\frac{y'}{\sqrt{1+(y')^2}} - \lambda x = c.$$

Now, using solution in the text

$$y_0' = -\frac{x-1}{\sqrt{2-(x-1)^2}},$$

and since $x \in [0,1]$,

$$\sqrt{1+(y_0')^2} = \sqrt{\frac{2}{2-(x-1)^2}}.$$

This gives us

$$\frac{y_0'}{\sqrt{1+(y_0')^2}} = \frac{1-x}{\sqrt{2}}.$$

Thus, y_0 satisfies the Euler–Lagrange equation if $\lambda = -c = -1/\sqrt{2}$. We also need to check the constraint, which is left for the reader!

8 *Troutman*, Problem 3.29.

Hint: The solution is a surprise!

Solution: Theorem 3.16 is about convex *functionals. When considering*

$$f(x, y, y') + \lambda g(y) = (y')^2 + \lambda y^2$$

it is therefore necessary to have $\lambda \geq 0$. However, the minimum for

$$\int_0^\pi \big(y'(x)^2 + \lambda y(x)^2\big)\,\mathrm{d}x$$

when $y(0) = y(\pi) = 0$ and $\lambda \geq 0$ is clearly obtained for $y \equiv 0$, such that the only solution Theorem 3.16 can provide is the trivial one:

$$\min \int_0^\pi y'(x)^2\,\mathrm{d}x$$

when

$$\int_0^\pi y(x)^2\,\mathrm{d}x = G(y_0) = 0.$$

There is no *$\lambda \geq 0$ to use in Theorem 3.16! (The actual solution of this problem is a challenge.)*