Thm

KKT conditions are sufficient under convexity

Assume:
- f - convex
- c_i - concave, $i \in \mathcal{I}$
- c_i - affine (linear), $i \in \mathcal{E}$
- x^* - KKT point for (P):

$$
(P) \quad \min f(x) \quad \text{s.t.} \left.\begin{cases} c_i(x) \geq 0, & i \in \mathcal{I} \\ c_i(x) = 0, & i \in \mathcal{E} \end{cases}\right\} =: \Omega
$$

Then x^* - global optimum ~~for~~ (P)

__Proof:__ We will show that $f(x) \geq f(x^*)$, $\forall x \in \Omega$

KKT - conditions for (P):

$$\nabla f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i(x^*) = 0$$

$$c_i(x^*) \geq 0, \quad i \in \mathcal{I}$$

$$c_i(x^*) = 0, \quad i \in \mathcal{E}$$

$$\lambda_i \geq 0, \quad i \in \mathcal{I}$$

$$\underbrace{\lambda_i \cdot c_i(x^*) = 0}_{\text{complementarity condition}}$$

Take $\forall x \in \Omega$

$$f(x) \geq f(x^*) + \nabla f^T(x^*)[x - x^*]$$ (f - convex, propositions in basic tools)

$$\nabla f(x^*)^T [x - x^*] =$$

$$= \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \nabla c_i^T(x^*)[x - x^*] \qquad (*)$$

- $\forall i \in \mathcal{E}$: $c_i(x) = c_i(x^*) = 0$, c_i - affine

$$\Rightarrow \nabla c_i^T(x^*)[x - x^*] = c_i(x) - c_i(x^*) = 0.$$

- $\forall i \in \mathcal{I}$: $c_i(x^*) > 0 \Rightarrow \lambda_i = 0$ (complementarity)
- $\forall i \in \mathcal{I}$: $c_i(x^*) = 0 \Rightarrow$

$$0 \leq c_i(x) = c_i(x) - c_i(x^*) \leq \nabla c_i^T(x^*)[x - x^*]$$

(c_i - concave)

$$\lambda_i \geq 0 \quad i \in \mathcal{I} \Rightarrow \lambda_i \nabla c_i^T(x^*)[x - x^*] \geq 0.$$

$$(*) \Rightarrow \quad \nabla f(x^*)^T[x - x^*] \geq 0.$$

$$\Rightarrow f(x) \geq f(x^*) \quad \forall \Omega$$

$\blacksquare$