Optimization Theory Project/2014

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1 Instructions

- The deadline for submitting the project is April 4 at 16:00.
- You can work in groups of 1–3 students. You can discuss the subject in larger groups, but when you sit down to write the report or the code, you should do so in small groups.
- Write your student number(s) on the reports, do not use names.
- Submit your reports (as PDF) and codes (ZIP) by e-mail to Lars.
- When you refer to books or other sources of information, always provide citations.
- You can write your project in English or Norwegian. Handwritten reports should be legible.
- In your report, provide concise answers to the questions posed in the project description. Additionally, provide sufficient details about the implementation (starting points, linesearch parameters, penalty parameter updates, etc). You may include code snippets of the most important parts of the algorithm, but do not provide listings of the full code. Finally, report on the algorithmic performance and try to relate what you observe to what is predicted by the theory.
- You are more than welcome to experiment with different algorithmic approaches to solving the problem, and compare them with what is suggested in this document. Correct and relevant extra work may to a degree compensate for deficiencies in completing other tasks in this project.
- The project counts for 20% of the course grade. If you do not do the project, you can still take the exam, but the best grade you can achieve is C.

2 Introduction

A *truss* is a mechanical structure constructed of straight elastic members (called *bars*) connected at joints (called *nodes*). Mechanical engineers have known truss structures for quite a while and can nowadays design very efficient structures based on mankind's acquired knowledge and experience. Many wooden and steel bridges, especially railroad bridges, are constructed as truss structures, see for example Skansen bridge (Fig. 1, left). Another example of a truss structure, which has become iconic, is Eiffel tower (Fig. 1, center). More commonly occurring truss towers are *transmission towers* (electricity pylons, see Fig. 1, right) utilized for supporting overhead power lines.

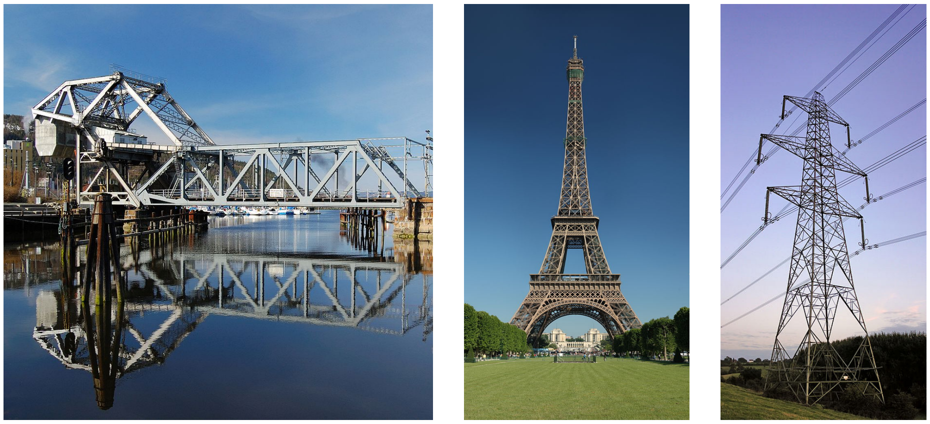

Fig. 1. Examples of truss structures. Left: Skansen railroad bridge; center: Eiffel tower; right: transmission tower.

In this project we will construct an optimization model and an efficient algorithm for solving it, which can automate the process of designing extremely efficient truss structures.

3 Mathematical model of a truss

3.1 Individual truss members

Consider a straight bar j connecting two distinct nodes i_1 and i_2 with coordinates $\boldsymbol{v}_{i_1} = (x_{i_1}, y_{i_1}, z_{i_1})^T$ and $\boldsymbol{v}_{i_2} = (x_{i_2}, y_{i_2}, z_{i_2})^T$. We arbitrarily choose the orientation of this bar as a vector originating at $\boldsymbol{v}_{i_1}$ and ending at $\boldsymbol{v}_{i_2}$. Let $\ell_j = \|\boldsymbol{v}_{i_2} - \boldsymbol{v}_{i_1}\|_2 > 0$ be the length of the bar, and $\boldsymbol{\tau}_j = (\boldsymbol{v}_{i_2} - \boldsymbol{v}_{i_1})/\ell_j$ be the vector of its directional cosines. Further let $q_j \in \mathbb{R}$ be an axial force acting on the bar when the truss structure is bearing a load — such as for example a train going across the bridge. The convention is that $q_j < 0$ corresponds to a compression force, when the end nodes of the bar are being pulled together, and $q_j > 0$ corresponds to a tensile force, when the bar ends are pulled apart. Finally, let the vectors $\boldsymbol{u}_{i_1}$ and $\boldsymbol{u}_{i_2}$ be the displacements (caused by the load on the truss structure) of the end nodes from their positions at rest. Assuming that the displacements are small, that is, that $\|\boldsymbol{u}_{i_1}\|_2/\ell_j \approx 0$ and $\|\boldsymbol{u}_{i_2}\|_2/\ell_j \approx 0$ the Hook's law states that the *stress* (force per area) in the bar is proportional to the *strain* (elongation per length) in the bar:

$$
\begin{aligned}
\text{stress} &= \text{Young's modulus} \times \text{strain}, \\
\frac{q_j}{A_j} &= E_j \frac{(\boldsymbol{u}_{i_2} - \boldsymbol{u}_{i_1}) \cdot \boldsymbol{\tau}_j}{\ell_j},
\end{aligned} \tag{1}
$$

where $A_j > 0$ is the cross-sectional area of the bar and $E_j > 0$ is the Young's modulus of the material, from which the bar is made.

3.2 Individual nodes

Consider an arbitrary node i in the truss structure. Let $\mathcal{J}_i^{\text{out}}/\mathcal{J}_i^{\text{in}}$ be the indices of the bars *originating*/*ending* at this node. The third Newton's law implies that at mechanical equilibrium, the sum of all forces at the node should be $\mathbf{0}$, that is:

$$-\sum_{j\in\mathcal{J}_i^{\text{out}}} q_j \boldsymbol{\tau}_j + \sum_{j\in\mathcal{J}_i^{\text{in}}} q_j \boldsymbol{\tau}_j = \boldsymbol{f}_i, \tag{2}$$

where $\boldsymbol{f}_i \in \mathbb{R}^3$ is the vector of *external* (with respect to the truss) forces acting at the node i. These forces may be:

- *unknown*, if the node i is *fixed* to an external structure/foundation, that is, $\boldsymbol{u}_i = \mathbf{0}$. In this case, $\boldsymbol{f}_i$ is the reaction force produced by the foundation;
- *prescribed* (often $\mathbf{0}$), if the node is *free* to move. In this case $\boldsymbol{f}_i$ is given by, for example, the weight of a train supported by the railroad bridge truss structure.

3.3 Full structure

Consider a truss with n nodes and m bars. We introduce vectors of axial forces $\boldsymbol{q} = (q_1, \ldots, q_m)^T \in \mathbb{R}^m$ and cross-section areas $\boldsymbol{A} = (A_1, \ldots A_m)^T \in \mathbb{R}^m$. We will also collect all the nodal displacements and external loads into vectors $\boldsymbol{u} = (\boldsymbol{u}_1^T, \ldots, \boldsymbol{u}_n^T)^T \in \mathbb{R}^{3n}$, $\boldsymbol{f} = ([\boldsymbol{f}_1]^T, \ldots, [\boldsymbol{f}_n]^T)^T \in \mathbb{R}^{3n}$. We will write $\boldsymbol{f}^{\text{supp}} \in \mathbb{R}^{3n_s}$, $\boldsymbol{f}^{\text{ext}} \in \mathbb{R}^{3(n-n_s)}$ to denote the subvectors of $\boldsymbol{f}$ of unknown support reaction forces/prescribed external forces, where n_s is the number of fixed nodes in the truss structure.

Let $\boldsymbol{B} \in \mathbb{R}^{3n\times m}$ be the following matrix:

$$B_{3(i-1)+k,j} = \begin{cases} -[\boldsymbol{\tau}_j]_k, & \text{if } j \in \mathcal{J}_i^{\text{out}} \ (i \text{ is the } \textit{first} \text{ node of bar } j), \\ [\boldsymbol{\tau}_j]_k, & \text{if } j \in \mathcal{J}_i^{\text{in}} \ (i \text{ is the } \textit{second} \text{ node of bar } j), \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, 3$, and $[\boldsymbol{\tau}_j]_k$ is the kth component of the vector $\boldsymbol{\tau}_j$. Similarly, let $\boldsymbol{I}_{\text{supp}} \in \mathbb{R}^{3n\times 3n_s}$, $\boldsymbol{I}_{\text{ext}} \in \mathbb{R}^{3n\times 3(n-n_s)}$ be two matrices with entries in $\{0, 1\}$ such that $\boldsymbol{f} = \boldsymbol{I}_{\text{supp}}\boldsymbol{f}^{\text{supp}} + \boldsymbol{I}_{\text{ext}}\boldsymbol{f}^{\text{ext}}$, $\boldsymbol{f}^{\text{supp}} = \boldsymbol{I}_{\text{supp}}^T\boldsymbol{f}$, and $\boldsymbol{f}^{\text{ext}} = \boldsymbol{I}_{\text{ext}}^T\boldsymbol{f}$. In this notation, equations (1) and (2) can be compactly stated as

$$\begin{aligned} \boldsymbol{D}\boldsymbol{q} &= \boldsymbol{B}^T\boldsymbol{u}, \\ \boldsymbol{B}\boldsymbol{q} &= \boldsymbol{I}_{\text{supp}}\boldsymbol{f}^{\text{supp}} + \boldsymbol{I}_{\text{ext}}\boldsymbol{f}^{\text{ext}}, \\ \boldsymbol{I}_{\text{supp}}^T\boldsymbol{u} &= \mathbf{0}, \end{aligned} \tag{4}$$

where $\boldsymbol{D} \in \mathbb{R}^{m\times m}$ is the diagonal matrix

$$\boldsymbol{D} = \begin{pmatrix} \frac{\ell_1}{E_1A_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\ell_m}{E_mA_m} \end{pmatrix}.$$

Question 1. Show that the system (4) constitutes the necessary and sufficient optimality conditions for the following problem:

$$\begin{aligned} \underset{(\boldsymbol{q},\boldsymbol{f}^{\text{supp}})}{\text{minimize}} \quad & \frac{1}{2}\sum_{j=1}^{m} \frac{\ell_j q_j^2}{E_jA_j}, \\ \text{subject to} \quad & \boldsymbol{B}\boldsymbol{q} = \boldsymbol{I}_{\text{supp}}\boldsymbol{f}^{\text{supp}} + \boldsymbol{I}_{\text{ext}}\boldsymbol{f}^{\text{ext}}. \end{aligned} \tag{5}$$

Question 2. Suppose that the feasible set of the problem (5) is non-empty. Show that it admits at least one solution. Conclude that the system (4) admits at least one solution under these assumptions.

Question 3. Suppose that the feasible set of the problem (5) is non-empty. Show that the problem admits exactly one solution. Hint: utilize the equivalence between the linear system (4) and the necessary and sufficient optimality conditions (5); show that $\boldsymbol{f}^{\text{ext}} = \boldsymbol{0}$ implies $(\boldsymbol{q}, \boldsymbol{f}^{\text{supp}}) = (\boldsymbol{0}, \boldsymbol{0})$ at the optimum, and that this in turn implies uniqueness of solutions to (5). Note: without further assumptions the vector $\boldsymbol{u}$ is not uniquely defined!

4 Optimal design of truss structures

4.1 Mathematical modelling

We will consider a simplified situation when the positions of the truss nodes and the bar materials are predetermined (thus ℓ_j, $\boldsymbol{\tau}_j$, and E_j, $j = 1, \ldots, m$ are given). Therefore we can only change the cross-sectional areas A_j.

We will study the following problem:

$$\begin{aligned} \underset{(\boldsymbol{A},\boldsymbol{q},\boldsymbol{f}^{\text{supp}})}{\text{minimize}} \quad & \frac{1}{2}\sum_{j=1}^{m} \frac{\ell_j q_j^2}{E_jA_j}, \\ \text{subject to} \quad & \boldsymbol{B}\boldsymbol{q} = \boldsymbol{I}_{\text{supp}}\boldsymbol{f}^{\text{supp}} + \boldsymbol{I}_{\text{ext}}\boldsymbol{f}^{\text{ext}}, \\ & \sum_{j=1}^{m} \rho_j \ell_j A_j \le M, \\ & \underline{A}_j \le A_j \le \overline{A}_j, \qquad j = 1, \ldots, m, \end{aligned} \tag{6}$$

where $M > 0$ is the maximal allowable mass of the truss structure, $\rho_j > 0$ is the density of the material in the bar j, $0 < \underline{A}_j < \overline{A}_j \le +\infty$, $j = 1, \ldots, m$, are lower and upper bounds on the bar cross-sectional areas. In problem (6) the objective is to minimize the elastic energy stored in the truss, which is equivalent to maximizing the *stiffness* of the structure (its ability to resist deformation in response to the applied force $\boldsymbol{f}^{\text{ext}}$).

Question 4. Show that the problem (6), whenever feasible, admits at least one optimal solution. Does the statement hold if we let $\underline{A}_j = 0$ and understand $q_j^2/0 = +\infty$, if $q_j \neq 0$ and 0, if $q_j = 0$?

Question 5. State the Karush–Kuhn–Tucker optimality conditions for the problem (6). Are they necessary and/or sufficient for the optimality (for the problem (6))? Motivate your answer.

4.2 Numerical algorithm

Consider the following barrier problem associated with the problem (6):

$$\begin{aligned} \underset{(\boldsymbol{A},\boldsymbol{q},\boldsymbol{f}^{\text{supp}})}{\text{minimize}} \quad & \frac{1}{2}\sum_{j=1}^{m} \frac{\ell_j q_j^2}{E_j A_j} - \mu \log\left[M - \sum_{j=1}^{m} \rho_j \ell_j A_j\right] \\ & \qquad - \mu \sum_{j=1}^{M} \log[(A_j - \underline{A}_j)(\overline{A}_j - A_j)], \\ \text{subject to} \quad & \boldsymbol{B}\boldsymbol{q} = \boldsymbol{I}_{\text{supp}}\boldsymbol{f}^{\text{supp}} + \boldsymbol{I}_{\text{ext}}\boldsymbol{f}^{\text{ext}}, \end{aligned} \tag{7}$$

where $\mu > 0$ is a small barrier parameter.

Question 6. State the Karush–Kuhn–Tucker optimality conditions for the problem (7). Compare them with the optimality conditions for the problem (6) and state the approximations for the Lagrange multipliers corresponding to the inequality constraints in (6) in terms of an optimal solution to (7). Are KKT conditions necessary and/or sufficient for the optimality (for the problem (7))? Motivate your answer.

Question 7. Implement a linesearch SQP algorithm for the barrier problem (7) based on the exact Hessian of the Lagrangian. Use ℓ_1 merit function and backtracking (Armijo) linesearch. Pay special attention to the linesearch procedure: the steplength may be bounded by the fact that the arguments of the logarithmic barrier should always be positive. Same warning applies to selecting the starting point for the algorithm.

4.3 Numerical tests

Test the algorithm on at least the examples described below. You may put $\rho = E = \overline{A}_j = 1$, $\underline{A}_j = 10^{-4}$, $M = 10$, $\mu = 10^{-4}$ in these examples. Visualize the optimal solutions by plotting the truss structure with lines, thickness of which is proportional to the square root of the computed cross-sectional areas.

1. Tower. Number of nodes: $n = 13$. Nodal coordinates: 12 nodes with coordinates in the set $\{\, (x, y, z) \mid x \in \{0,1\}, y \in \{0,1\}, z \in \{0,1,2\} \,\}$ and one node with coordinates $(0.5, 0.5, 2.5)$. Fixed nodes: 4 nodes “on the ground” (coordinate $z = 0$). Connected nodes: all distinct nodes (i_1, i_2), which are at most $\sqrt{3}$ apart from each other, are connected with a bar (total of $m = 58$ bars). External load: the only non-zero prescribed external load $(0, 0, -1)$ is applied at the “top” node (with coordinates $(0.5, 0.5, 2.5)$).

2. Cantilever: same as example 1, but the external load is $(1, 0, 0)$.
3. Bridge. Number of nodes: $n = 20$. Nodal coordinates: set $\{ (x, y, z) \mid x \in \{0, 1, 2, 3, 4\}, y \in \{0, 1\}, z \in \{0, 1\} \}$. Fixed nodes: 4 nodes with coordinates $\{ (x, y, z) \mid x \in \{0, 4\}, y \in \{0, 1\}, z = 0 \}$. Connected nodes: all distinct nodes (i_1, i_2), which are at most $\sqrt{3}$ apart from each other, are connected with a bar. Total number of bars: $m = 94$. External load: all free nodes "at the bottom of the bridge", that is, with coordinate $z = 0$ are loaded with an external load $(0, 0, -1)$; at the rest of the free nodes the external load is $\mathbf{0}$.

Feel free to come up with more, perhaps larger, examples of your own and test the algorithm on those.