TMA 4195 Mathematical Modelling

December 12, 2007

Solution with additional comments

1 Problem

The air resistance, F, of a car depends on its length, L, cross sectional area, A, its speed relative to the air, U, the density of the air, ρ, and the air's kinematic viscosity, ν.

(a) *Use dimensional analysis to derive the equation*

$$F = \rho U^2 A \phi \left(\frac{UL}{\nu}, \frac{A}{L^2} \right). \tag{1}$$

In order to determine the function ϕ, the engineers have suggested to test 1:10 scale models in the long water tank at the Tyholt model basin by dragging them through water (the tank has $5 \times 10m$ cross section, and a $270m$ length). The scale model is 1/10 of the original's size.

(b) *Is this a good idea?*

(For estimates: Air: $\nu = 10^{-5}\mathrm{m}^2/\mathrm{s}$ *and* $\rho = 1\mathrm{kg}/\mathrm{m}^3$. Water: $\nu = 10^{-6}\mathrm{m}^2/\mathrm{s}$ *and* $\rho = 10^3\mathrm{kg}/\mathrm{m}^3$).

Solution:

(a) The formula gives us all the physical quantities we need, and, using the formula for determining any unknown units, we can state the dimensions matrix:

	F	U	L	A	ρ	ν
m	1	1	1	2	-3	2
s	-2	-1	0	0	0	-1
kg	1	0	0	0	1	0

(2)

The matrix has rank 3, and with U, L and ρ as *core variables,* we obtain

$$\begin{aligned} \pi_1 &= \frac{A}{L^2}, \\ \pi_2 &= \frac{\nu}{UL}, \\ \pi_3 &= \frac{F}{\rho U^2 L^2}. \end{aligned} \tag{3}$$

This gives the formula if we transform $\Psi(\pi_1, \pi_2, \pi_3) = 0$ into

$$\pi_3 = \pi_1 \phi \left(\frac{1}{\pi_2}, \pi_1 \right). \tag{4}$$

(Dimension analysis never returns a unique expression, and in principle all reasonable ways to write the formula are equally good. Instead of L we could have used A or $\sqrt{A}$ as core variables).

(b) The model will reduce L with a factor 10 and the area with a factor 100 (It would probably be better to say 10:1 if one is thinking of "original $\to$ model"). In order to map the function ϕ, π_1 and π_2 should have about the same values as the original. This is fine for π_1, which is unchanged. For $\pi_2^{-1} = UL/\nu$ (Reynolds number) to be the same, we need that $U_{\text{model}} = U_{\text{original}}$ since L/ν is not changed between from the original in the air and the model in water. For the force on the model, $(\rho_{\text{water}} L^2_{\text{model}}) = 10 \times \left(\rho_{\text{air}} L^2_{\text{original}}\right)$. Apart from that, it is clear that the test present big technical difficulties. For example, around 100 km/h the model uses less than 10 seconds along the whole tank's length. The dragging machinery cannot reach this speed, so one can forget about testing the model for speeds where air resistance is relevant. Finally, the model should be rolling along the bottom. No matter how reasonable this all could seem, I have never heard about someone trying it.

Digression: For realistic speeds, ϕ is not very influenced by the Reynolds number, but depends heavily on the cars streamlined geometry. The car industry uses the equation

$$F_D = \rho U^2 A \frac{C_d}{2}, \tag{5}$$

where C_d is the so-called *drag-coefficient*. On modern cars, it has a value around 0.3.

2 Problem

Determine the equilibrium points and whether they are stable or unstable for the following equation:

$$\frac{du}{dt} = \left(u - u^2\right)(u - \mu), \; u \geq 0, \mu \geq 0. \tag{6}$$

Solution:

This is a standard problem where the equilibrium solutions can be found by solving

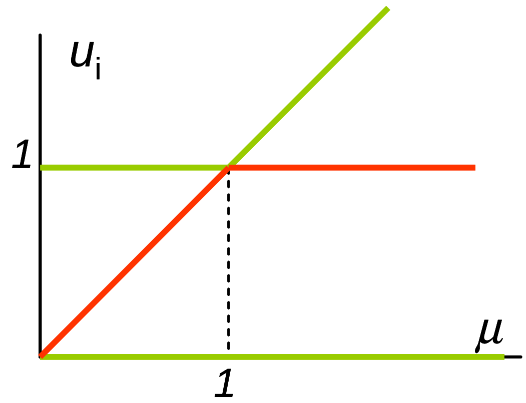
$$f(u_0, \mu) = 0. \tag{7}$$

The stability is decided by studying the Taylor expansion around u_0 and setting $u = u_0 + y + HOT$:

$$\frac{dy}{dt} = \frac{\partial f}{\partial u}(u_0, \mu)\, y + HOT \tag{8}$$

Stability can normally be decided by using the sign of $\frac{\partial f}{\partial u}(u_0, \mu)$. In this case, the equilibrium solutions are obvious:

$$\begin{aligned} u_0 &= 0, \\ u_1 &= 1, \\ u_2 &= \mu. \end{aligned} \tag{9}$$



Figur 1: Bifurcation diagram for Problem 2. Red is unstable and green is stable.

Moreover,

$$\begin{aligned}\frac{\partial f}{\partial u}(u_0,\mu) &= -\mu, \\ \frac{\partial f}{\partial u}(u_1,\mu) &= \mu-1, \\ \frac{\partial f}{\partial u}(u_2,\mu) &= \left(\mu-\mu^2\right).\end{aligned} \tag{10}$$

Therefore, we can make the bifurcation diagram as in Fig. 1 (for $u \geq 0$, $\mu \geq 0$).

To be complete, one should also check $u=0$ for $\mu=0$, and $u=1$ for $\mu=1$. For the first point, the equation for the perturbation is $\dot{y}=y^2+y^3$, and the point is clearly *unstable* if $y(0)>0$.

At the point $u=1$, $\mu=1$, the equation becomes $\dot{y}=-y^2-y^3$, which is *stable* for $y(0)>0$ and *unstable* for $y(0)<0$ and small, as it should be expected from the diagram.

3 Problem

The cell density, n^*, in a part of the body may be modelled as

$$\frac{dn^*}{dt^*}=\alpha n^*-\omega n^*, \tag{11}$$

where α is the birth rate and ω the death rate. In order to prevent that the density runs astray, the cells produce a so-called inhibitor which dampens uncontrolled growth. The inhibitor has density c^* and works by changing the the birth rate to

$$\alpha=\frac{\alpha_0}{1+c^*/A}. \tag{12}$$

The production of the inhibitor is proportional with n^*, while it breaks down with rate δ:

$$\frac{dc^*}{dt^*}=\beta n^*-\delta c^*. \tag{13}$$

This system has a time scale ω^{-1} connected to the breakdown of the cells, and a time scale δ^{-1} connected to the breakdown of the inhibitor. It is known that $\omega^{-1} \gg \delta^{-1}$.

(a) Scale the system by applying ω^{-1} as the time scale and A as a scale for c^*. Show that the system with a certain scale for n^* may be written

$$\begin{aligned} \dot{n} &= \left(\frac{\kappa}{1+c} - 1\right) n, \\ \varepsilon \dot{c} &= n - c. \end{aligned} \tag{14}$$

What is the meaning of ε and κ? What may be said about the size of ε, and what is such a system called? Determine what kind of equilibrium point the trivial equilibrium point $(0,0)$ is. (Here and below we assume that κ is somewhat larger than 1).

(b) Determine the path and the equation of the motion for the outer solution of Eqn. (14) to leading order ($O(1)$). Show, without necessarily solving the differential equation, that all motion on this path converges to an equilibrium point which also is an equilibriums point for the full system.

(c) Determine to leading order the inner solution of (14) by introducing a new time scale. Then determine a uniform, approximate solution (It is not possible to solve the equation in (b) explicitly).

Solution:

(a) As suggested, we use the time scale $T = 1/\omega$, and C_0 as a scale for c^*. Comparing Eqns. 13 and 14, we find that we must choose a scale for n^* as

$$N_0 = \frac{\delta}{\beta} C_0, \tag{15}$$

If this is used, the equation 14 follows immediately. The remaining parameters are

$$\begin{aligned} \kappa &= \frac{\alpha_0}{\omega} = \frac{1/\omega}{1/\alpha_0}, \\ \varepsilon &= \frac{\omega}{\delta} = \frac{1/\delta}{1/\omega}. \end{aligned} \tag{16}$$

Both are relations between time scales, where κ is said to be bigger than 1, while $0 < \varepsilon \ll 1$. This is, therefore, a singularly perturbed system. The equilibrium solutions follow be putting the RHS to 0:

$$\begin{aligned} (n_1, c_1) &= (0, 0), \\ (n_2, c_2) &= (\kappa - 1, \kappa - 1) \end{aligned} \tag{17}$$

Linearizing around $(0,0)$ gives

$$\begin{bmatrix} \kappa - 1 & 0 \\ \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \end{bmatrix}. \tag{18}$$

Since the eigenvalues are $\lambda_1 = \kappa - 1 > 0$ and $\lambda_2 = -1/\varepsilon < 0$, the point is a saddle point (It was not required analyze the other equilibrium point).

(b) Equation 14 is a singular perturbed system since $\varepsilon \ll 1$ and we write $n(t) = n_0(t) + \varepsilon n_1(t) + \cdots$, and similarly for c for the outer solution. The leading order is $n_0(t)$ and $c_0(t)$, and we first find that $n_0(t) = c_0(t)$. This gives us the equation

$$\frac{dn_0}{dt} = \left(\frac{\kappa}{1+n_0} - 1\right) n_0. \tag{19}$$

The point $(\kappa - 1, \kappa - 1)$ is still an equilibrium point, and from the sign test we see that $(\kappa - 1)$ is stable for 19, no matter where we choose to start for $0 < n_0(0) < \infty$.

It is almost possible to solve the equation 19 since we can write

$$\frac{1+n_0}{(\kappa - 1 - n_0)\, n_0} dn_0 = dt, \tag{20}$$

or, since $\kappa > 1$,

$$\left(\frac{1}{n_0} - \frac{\kappa}{n_0 - \kappa + 1}\right) dn_0 = \frac{dt}{\kappa - 1}. \tag{21}$$

Thus, an *implicit* solution is given by

$$\frac{n_0}{|n_0 - (\kappa - 1)|^{\kappa}} = e^{(t-t_0)/(\kappa-1)}. \tag{22}$$

Since $e^t \to \infty$ for $t \to \infty$, we have that $n_0(t) \underset{t\to\infty}{\longrightarrow} \kappa - 1$.

(c) As usual, we try to scale the time as $\tau = t/\varepsilon$ for the initial phase of the motion. This gives us the following equations, where we use $N(\tau)$ and $C(\tau)$ to separate the inner solution from the outer:

$$\begin{aligned} \frac{dN}{d\tau} &= \varepsilon \left(\frac{\kappa}{1+C} - 1\right) N, \\ \frac{dC}{d\tau} &= N - C. \end{aligned} \tag{23}$$

To the leading order,

$$\begin{aligned} \frac{dN_0}{d\tau} &= 0, \\ \frac{dC_0}{d\tau} &= N_0 - C_0, \end{aligned}$$

which gives us

$$\begin{aligned} N_0(\tau) &= n(0), \\ C_0(\tau) &= [c(0) - n(0)]\, e^{-\tau} + n(0). \end{aligned} \tag{24}$$

For the outer solution, we do not have a starting point, but we know from (b) that $n_0(t) = c_0(t)$. Let us assume that

$$\begin{aligned} n_0(0) &= A, \\ c_0(0) &= A. \end{aligned} \tag{25}$$

The "matching-condition" in the most simple form is now

$$\lim_{\tau \to \infty} C_0(\tau) = \lim_{t \to 0} c_0(t), \quad \lim_{\tau \to \infty} N_0(\tau) = \lim_{t \to 0} n_0(t), \tag{26}$$

and fortunately, $n_0(0) = A$ satisfies both conditions. The known uniform solution is

$$u_u(t) = u_0(t) + U_0(\tau) - \lim_{\tau \to \infty} U_0(\tau). \tag{27}$$

Therefore, we obtain

$$n_u(t) = n_0(t), \; n_0(0) = n(0), \quad c_u(t) = n_0(t) + [c(0) - n(0)] e^{-t/\varepsilon}. \tag{28}$$

In general,

$$\dot{c} = \frac{1}{\varepsilon}(n - c) \tag{29}$$

tells us that $\dot{c}$ becomes big and positive whenever $n - c \gg \mathcal{O}(\varepsilon)$, big and negative whenever $n - c \gg \mathcal{O}(-\varepsilon)$. Moreover, we see that $\dot{n} < 0$ if $c > \kappa - 1$, and $\dot{n} > 0$ if $c < \kappa - 1$. Without what we found in (b), this gives a good qualitative insight of trajectories that have been numerically computed in fig. 2. It is obvious that, as long at we do not start at $n(0) = 0$, we will, for $t \to \infty$, end in the equilibrium point $(\kappa - 1, \kappa - 1)$.

4 Problem

(a) *Define the scaled flux and kinematic velocity in the standard model for road traffic, which leads to the differential equation:*

$$\rho_t + (1 - 2\rho)\rho_x = 0. \tag{30}$$

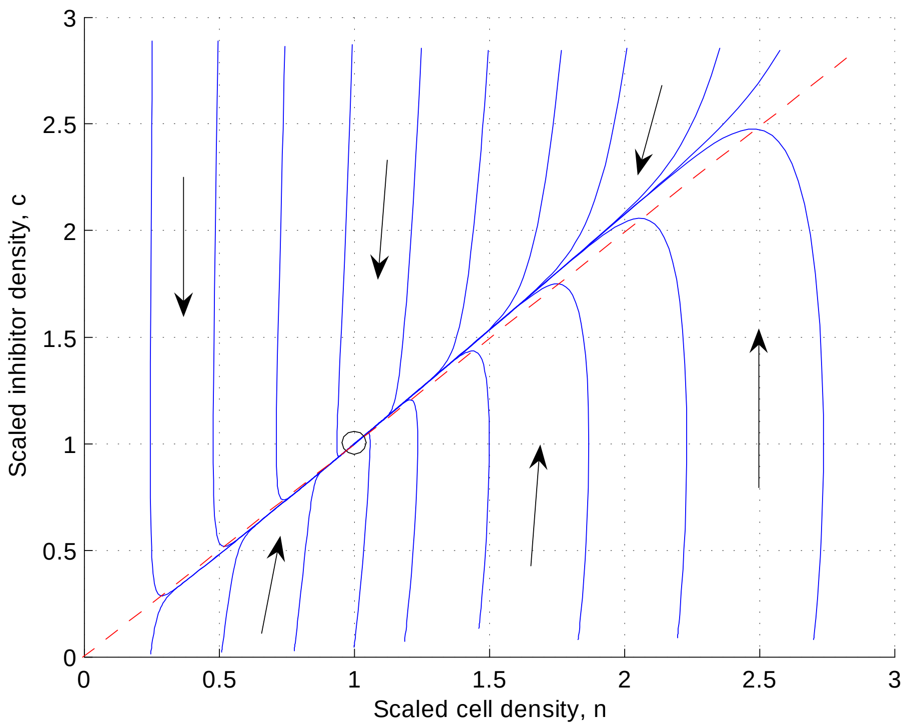
Sketch the characteristics and the solution $\rho(x,t)$ to Eqn. (30) for $t > 0$ if

$$(i) \quad \rho(x,0) = \begin{cases} 1 & x < 0, \\ 0 & x \geq 0. \end{cases} \qquad (ii) \quad \rho(x,0) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases} \tag{31}$$

We are from now considering a situation where cars are continuously entering and leaving the road (the road itself is a one-way street). This will be modelled as a source/sink term, such that the complete equation becomes

$$\rho_t + (1 - 2\rho)\rho_x = \varepsilon\left(\frac{1}{2} - \rho\right), \; \varepsilon > 0. \tag{32}$$

(If $\rho < \frac{1}{2}$, there is a net influx of cars, whereas cars are leaving the road if $\rho > \frac{1}{2}$).



Figur 2: Numerical solutions of the scaled system for $\kappa = 2$ and $\varepsilon = 0.1$. The red dashed line is the path of the first order outer solution.

(b) *Show that a characteristic curve starting at* $(0, x_0, \rho_0)$ *may, for* $t \geq 0$*, be written as*

$$\left\{ t \ , \ x_0 + \frac{(1-2\rho_0)}{\varepsilon}\left(1 - e^{-\varepsilon t}\right) \ , \ \frac{1}{2} + \left(\rho_0 - \frac{1}{2}\right) e^{-\varepsilon t} \right\}. \tag{33}$$

(c) *Find the solution of Eqn. (32) for* $t > 0$ *with (i) in (31) as initial condition.*

(d) *Show that the solution of Eqn. (32) for* $t > 0$ *with (ii) in (31) as initial condition develops a shock. Use the conservation law to argue that the location of the shock may be stationary. Assuming this, determine the solution.*

(*Hint: The equation*

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} - R(x,y,z) = 0$$

has the following equations for the characteristics

$$\frac{dx}{ds} = P(x,y,z), \frac{dy}{ds} = Q(x,y,z), \frac{dz}{ds} = R(x,y,z) \ ).$$

Solution:

(a) In the standard model, the density ρ is the number of cars per unit of length. It is scaled so as $0 \leq \rho \leq 1$. The speed of the cars is $v = 1 - \rho$, whereas the flux is $j = \rho(1-\rho)$, and the kinematic velocity $c(\rho) = \frac{dj}{d\rho} = 1 - 2\rho$. We shall find the solution for two different cases, and, for sake of clarity, it is nice to make two sketches as in Fig. 3. The situation (i) leads to a rarefaction wave, while situation (ii) leads to a shock. Note that we have used that, for the standard model, the shock speed is given by $U = 1 - \rho_1 - \rho_2$, and therefore, $U = 0$ for the problem (ii). Consequently, the solution becomes simply

$$\rho(x,t) = \rho(x,0). \tag{34}$$

For the rarefaction wave, ρ is piecewise linear, i.e.

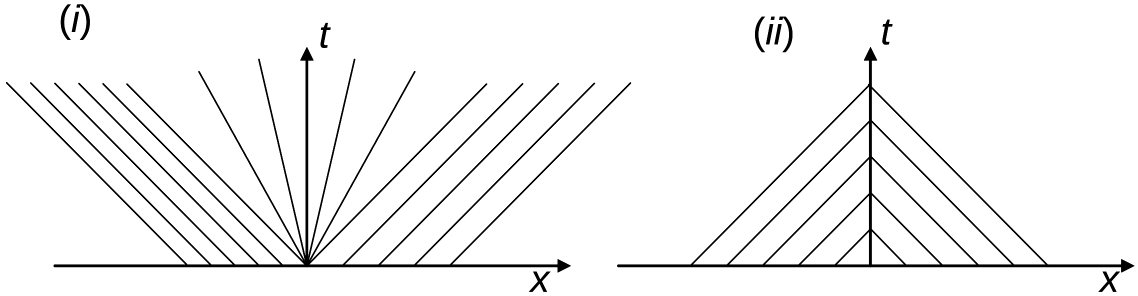
$$\rho(x,t) = \begin{cases} 1, & x \leq -t, \\ \frac{1}{2}\left(1 - \frac{x}{t}\right), & -t \leq x \leq t, \\ 0, & x \geq t. \end{cases} \tag{35}$$

(b) The equations for the characteristics (using s as parameter) may be written

$$\begin{aligned} \frac{dt}{ds} &= 1, \\ \frac{dx}{ds} &= 1 - 2\rho, \\ \frac{d\rho}{ds} &= -\varepsilon\left(\rho - \frac{1}{2}\right). \end{aligned} \tag{36}$$

Since the first equation gives $t = A + s$, we can choose $s = t$. Thus, the third equation becomes

$$\frac{d\rho}{\rho - \frac{1}{2}} = -\varepsilon dt, \tag{37}$$



Figur 3: Situation (i) leads to a rarefaction wave, whereas (ii) gives a shock.

with general solution

$$\ln\left|\rho-\frac{1}{2}\right|=C_1-\varepsilon t, \tag{38}$$

or

$$\rho=\frac{1}{2}+C_2e^{-\varepsilon t}. \tag{39}$$

(It is even simpler to observe that the equation may be written $\dot{\rho}+\varepsilon\rho=\frac{\varepsilon}{2}$ and the general solution is $\rho=C_2e^{-\varepsilon t}+\frac{1}{2}$)
Here $\rho(0)=\rho_0$, and consequently

$$\rho=\frac{1}{2}+\left(\rho_0-\frac{1}{2}\right)e^{-\varepsilon t}. \tag{40}$$

Finally, we compute x from

$$\frac{dx}{ds}=1-2\rho=-\left(2\rho_0-1\right)e^{-\varepsilon t}, \tag{41}$$

i.e.

$$x=C_3+\frac{2\rho_0-1}{\varepsilon}e^{-\varepsilon t}, \tag{42}$$

and with $x(0)=x_0$,

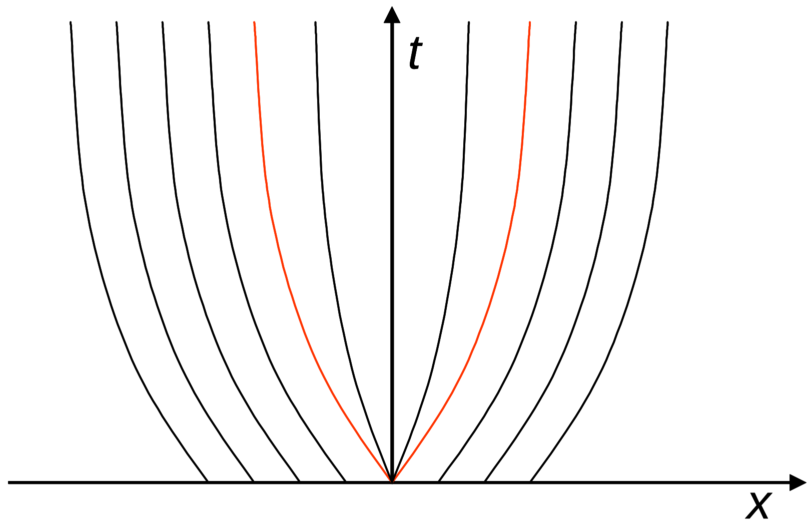
$$x=x_0+\frac{2\rho_0-1}{\varepsilon}\left(e^{-\varepsilon t}-1\right)=x_0+\frac{1-2\rho_0}{\varepsilon}\left(1-e^{-\varepsilon t}\right), \tag{43}$$

as was given in the exercise's text.

(c) Here we use the characteristics from (b):

$$\left\{t\ ,\ x_0+\frac{1-2\rho_0}{\varepsilon}\left(1-e^{-\varepsilon t}\right)\ ,\ \frac{1}{2}+\left(\rho_0-\frac{1}{2}\right)e^{-\varepsilon t}\right\}. \tag{44}$$

From the projection of the characteristics in the (x,t)-plane, we see that if we start in x_0 with value ρ_0 for $t=0$, the direction from $(x_0,0)$ will be $dx/dt=1-2\rho_0$. Besides, $\lim_{t\to\infty}dx/dt=0$, and $\lim_{t\to\infty}x(t)=x_0+(1-2\rho_0)/\varepsilon$, as shown in the sketch (Fig. 4).



Figur 4: Sketch of the characteristics for problem (i). The characteristics are not influenced outside the rarefacton wave.

It is not difficult to conclude that we obtain a rarefaction wave as shown in the sketch. Note also that $\lim_{t\to\infty}\rho(t) = 1/2$. We can also note that the solution outside the limit characteristics

$$x(t) = \pm\frac{1-e^{-\varepsilon t}}{\varepsilon} \tag{45}$$

develop independent of the rarefaction wave, and here,

$$\rho(x,t) = \frac{1}{2} + \left(\rho_0 - \frac{1}{2}\right)e^{-\varepsilon t}, \;\; \rho_0 = 0, 1. \tag{46}$$

Within the expansion fan, the solution at (x,t) is given implicitly by first choosing ρ_0 from

$$0 + \frac{1-2\rho_0}{\varepsilon}\left(1-e^{-\varepsilon t}\right) = x, \tag{47}$$

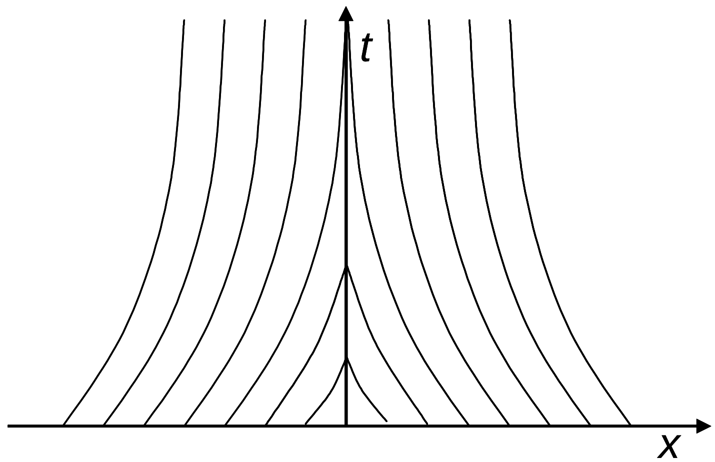
i.e.

$$\rho_0 = \frac{1}{2}\left(1 - \frac{\varepsilon x}{1-e^{-\varepsilon t}}\right). \tag{48}$$

Thus,

$$\begin{aligned}\rho(x,t) &= \frac{1}{2} + \left(\rho_0 - \frac{1}{2}\right)e^{-\varepsilon t} \\ &= \frac{1}{2}\left(1 - \frac{\varepsilon x e^{-\varepsilon t}}{1-e^{-\varepsilon t}}\right)\end{aligned} \tag{49}$$

(d) If we look at the situation in (c), the characteristics that start on each side of (and close enough to) the origin will collide, as shown in fig. 5.



Figur 5: The characteristics near the origin will collide for problem (ii). The graph shows a stationary shock as argued for in the text.

A stationary shock in the origin gives a symmetric situation where the ρ-values on each side of $x = 0$ is symmetric with respect to $\rho = 1/2$. This gives a continuous flux over the shock since $j = \rho(1-\rho)$. The source term does not contribute if the width of the control volume goes to 0. Besides, a control volume that is symmetric with respect to $x = 0$, does not have a net change in the content, since the source function is antisymmetric with respect to $\rho = 1/2$. Assuming that the shock is stationary, we then obtain the solution:

$$\rho(x,t) = \begin{cases} \frac{1}{2}\left(1 - e^{-\varepsilon t}\right) & x < 0, \\ \frac{1}{2}\left(1 + e^{-\varepsilon t}\right) & x > 0. \end{cases} \tag{50}$$