	F	U	L	W	D	ρ	ν	g
m	1	1	1	1	1	-3	2	1
s	-2	-1	0	0	0	0	-1	-2
kg	1	0	0	0	0	1	0	0

Table 1: Dimension matrix

TMA 4195 Matematisk modellering
Exam Tuesday December 16, 2008
09:00 – 13:00
Problems and solution with additional comments

1 Problem

The necessary force (F) to keep a ship at a constant speed (U) depends on its shape; primarily the length (L), width (W), and its depth into the water (D). In addition, the water density, ρ, the water viscosity, ν, and the acceleration of gravity, g, play a part.

Use dimensional analysis to find an expression for the force which includes the two most famous dimensionless numbers in ship design:

$$\text{Froude number: } Fr = U/\sqrt{Lg},$$
$$\text{Reynolds number: } Re = LU/\nu.$$

Ideally, a scale model[1)] of the ship should be tested experimentally in water by keeping the dimensionless numbers for the model equal to those of the original ship. Is this really possible?

Hints: $[F] = \text{kgm/s}^2$, $[\rho] = \text{kg/m}^3$, $[\nu] = \text{m}^2/\text{s}$, $[g] = \text{m/s}^2$.

[1)] *A scale model is a model of the ship with the same geometric shape, but with a smaller size (Say, L =1m for the model, compared to 200m for the original ship).*

Solution:

Using the information provided in the problem, we assume

$$F = f\left(U, L, W, D, \rho, \nu, g\right). \tag{1}$$

The dimension matrix follows immediately and is shown in Table 1.

The rank is 3, and there are several possibilities for core variables (avoiding F): (U, L, ρ), (g, D, ρ), (ν, ρ, W), $\cdots$. However, if one aims for the Froude and Reynolds numbers, the choice (U, L, ρ) looks reasonable. With 8 variables, there are $8 - 3 = 5$ dimensionless combinations.

Since Re involves ν and Fr involves g, it is easy to arrive at the formula

$$F = \rho U^2 L^2 \times \Phi\left(Re, Fr, \frac{W}{L}, \frac{D}{L}\right). \tag{2}$$

The scale model keeps W/L and D/L unchanged, so if we forget those, we need to map the function

$$\pi_1 = \frac{F}{\rho U^2 L^2} = \Phi\left(Re, Fr\right) \tag{3}$$

for ranges of Re and Fr typical for the original ship. Assume that length of the model, L_m, is equal to rL, where r is about 10^{-2}. If we aim to keep Fr, we have to run the model with $U_m = \sqrt{r}U$, which looks feasible. However, for the same ν, the Reynolds number would then be a factor $r^{3/2}$ off. The only way to compensate this would actually be find a fluid with a correspondingly small viscosity, but this does not exist. If we start by keeping the Reynold number (rather unrealistic!) we run into similar problems.

Scale testing of ships (e.g. at the Tyholt wave basins), is carried out by keeping the Froude number (related to wave induced resistance) and neglecting the variation in the Reyleigh number. The first analysis of this situation dates back to one of the founders of Fluid Mechanics, Lord Kelvin. Read more about *ship resistance* on the Internet.

NB! This problem has many different solutions, all equally good as long as the set of variables is the same, and the core variables are chosen properly.

2 Problem

A chemical reactor consists of a tank filled with a solid catalyst. A fluid containing a dissolved chemical with concentration c_I flows into the tank. The fluid flow Q is measured in volume per second and the available volume for the fluid inside the reactor is V. Some of the chemical is converted into a product by the catalyst and this product leaves the tank with the fluid stream. The concentration of non-converted chemical in the tank is $c^*(t^*)$, and this is also the concentration in the fluid leaving the tank.

No volume changes are involved in these reactions, and the reactor is well-mixed with no significant concentration gradients inside.

The catalyst's efficiency is measured in terms of a quantity a^*, and the amount of chemical converted per time and volume unit is a^*c^*. Moreover, the change in a^* per time unit is $-k_d c^* a^*$. At the start of reaction, $t^* = 0$, we have $a^*(0) = a_I$ and $c^*(0) = c_I$.

(a)

Formulate the equations for the concentration of the chemical and the catalyst's efficiency, and derive the time scales

$$T_1 = \frac{1}{a_I}, \tag{4}$$

$$T_2 = \frac{1}{k_d c_I}, \tag{5}$$

What are these scales expressing?

In the following it is known that $T_1 \ll T_2$.

(b)

Use one of the time scales, write $t^* = \tau T$, and scale the equations so that

$$\begin{aligned} \frac{dc}{d\tau} &= -ca + \alpha\left(1-c\right), \\ \frac{da}{d\tau} &= -\varepsilon ca, \\ c\left(0\right) = 1,\ & a\left(0\right) = 1,\ \tau \geq 0. \end{aligned} \tag{6}$$

Explain the meaning of ε and α.

Determine the functions C_0 and A_0 in the perturbation expansions $c\left(\tau\right) = C_0\left(\tau\right) + \varepsilon C_1\left(\tau\right) + \cdots$ and $a\left(\tau\right) = A_0\left(\tau\right) + \varepsilon A_1\left(\tau\right) + \cdots$.

(NB! Here and in the following two points we assume for simplicity that $\alpha = 1$).

(c)

Use the other time scale and show that this leads to a singular perturbation system with Eqn. 6 as the "inner" system.

Determine the "outer" solution to leading order, i.e., find a_0 and c_0 in the perturbation expansions $c\left(t\right) = c_0\left(t\right) + \varepsilon c_1\left(t\right) + \cdots$ and $a\left(t\right) = a_0\left(t\right) + \varepsilon a_1\left(t\right) + \cdots$.

(Hint: Derive and solve a separable differential equation for a_0, which, however, leads to an implicit expression for the actual solution).

(d)

Carry out a matching of the inner and outer solutions above and state leading order uniform solutions.

Solution:

(a) The conservation law for the chemical may be expressed in differential form as

$$\frac{d\left(Vc^*\right)}{dt^*} = -Va^*c^* + Qc_I - Qc^*. \tag{7}$$

(All terms express change in mass per time unit). For the efficiency we have similarly

$$\frac{da^*}{dt^*} = -k_d c^* a^*. \tag{8}$$

The time scale T_1 is derived from the decay of the chemical in a closed volume at the beginning of the reaction,

$$\begin{aligned} \frac{dc^*}{dt^*} &= -a_I c^*, \\ &\Downarrow \\ c^* &= C\exp\left(-t^*/\left(1/a_I\right)\right). \end{aligned} \tag{9}$$

The time scale T_2 relates to the decay in the efficiency of the catalyst,

$$\frac{da^*}{dt^*} = -k_d c_I a^*. \tag{10}$$

There is also a third time scale, $T_3 = V/Q$. This is simply the time it takes to fill a volume V with the flow Q.

The most elegant derivation of the time scales is to follow Lin&Segel's recipe:

$$T_1 = \frac{\max|c^*|}{\max|dc^*/dt^*|} = \frac{|c_I|}{|c_I a_I|} = \frac{1}{a_I},$$
$$T_2 = \frac{\max|a^*|}{\max|da^*/dt^*|} = \frac{|a_I|}{|k_d c_I a_I|} = \frac{1}{k_d c_I}.$$

(b) We consider the following scales

$$c^* = c_I c,\ \ a^* = a_I a,\ \ t^* = T\tau. \tag{11}$$

This leads to

$$\begin{aligned}\frac{1}{a_I T}\frac{dc}{d\tau} &= -ac + \frac{Q}{a_I V}(1-c), \\ \frac{1}{T k_d c_I}\frac{da}{d\tau} &= -ca.\end{aligned} \tag{12}$$

It now obvious that T should be equal to T_1, and

$$\varepsilon = \frac{1/a_I}{1/(k_d c_I)} = \frac{T_1}{T_2}. \tag{13}$$

Moreover,

$$\alpha = \frac{T_1}{T_3}. \tag{14}$$

Since $T_1 \ll T_2$, ε is a small dimensionless parameter, and Eqns. 6 represents a regular perturbation problem. Now, $\alpha = 1$, and the system is

$$\begin{aligned}\frac{dc}{d\tau} &= -ca + (1-c), \\ \frac{da}{d\tau} &= -\varepsilon ca.\end{aligned} \tag{15}$$

Inserting the power series in ε, we obtain to the leading order

$$\begin{aligned}\frac{dC_0}{d\tau} &= -C_0 A_0 + (1-C_0), \\ \frac{dA_0}{d\tau} &= 0.\end{aligned} \tag{16}$$

In addition, the initial conditions, which may be satisfied exactly, are

$$C_0(0) = 1,\ \ A_0(0) = 1. \tag{17}$$

This leads to $A_0(\tau) = 1$, and

$$\frac{dC_0}{d\tau} + 2C_0 = 1, \tag{18}$$

that is,

$$C_0(\tau) = \frac{1}{2}\left(1 + e^{-2\tau}\right). \tag{19}$$

(c) By changing the time scale to $T = T_2$ in Eqns. 12, and using $t^* = T_2 t$, we obtain

$$\begin{aligned} \varepsilon\frac{dc}{dt} &= -ac + (1 - c), \\ \frac{da}{dt} &= -ca. \end{aligned} \tag{20}$$

which is a typical singular perturbation system with a small parameter in front of the highest derivative. We also observe that $\tau = t/\varepsilon$, in accordance with Eqns. 6 being the "inner" system of Eqn. 20. Inserting $c(t) = c_0(t) + \varepsilon c_1(t) + \cdots$ and $a(t) = a_0(t) + \varepsilon a_1(t) + \cdots$ gives to leading order

$$\begin{aligned} 0 &= -a_0 c_0 + (1 - c_0), \\ \frac{da_0}{dt} &= -c_0 a_0. \end{aligned} \tag{21}$$

From the first equation we obtain

$$c_0 = \frac{1}{1 + a_0}, \tag{22}$$

and hence,

$$\frac{da_0}{dt} = -\frac{a_0}{1 + a_0}. \tag{23}$$

This is a separable equation which may be written

$$\left(\frac{1}{a_0} + 1\right) da_0 = -dt, \tag{24}$$

and integrated to

$$\ln a_0 + a_0 = -t + t_0, \tag{25}$$

where t_0 is an unknown constant. It is not possible to express a_0 as an elementary function of t.

(d) We have now obtained the inner solution

$$\begin{aligned} A_0(\tau) &= 1, \\ C_0(\tau) &= \frac{1}{2}\left(1 + e^{-2\tau}\right), \end{aligned} \tag{26}$$

which also fulfil the initial conditions, and the outer solution,

$$\begin{aligned} c_0(t) &= \frac{1}{1 + a_0(t)}, \\ \ln a_0(t) + a_0(t) &= -t + t_0 \end{aligned} \tag{27}$$

with one unknown constant t_0.

The matching principle in its simplest form requires that

$$\lim_{\tau\to\infty} C_0(\tau) = \lim_{t\to 0} c_0(t), \\ \lim_{\tau\to\infty} A_0(\tau) = \lim_{t\to 0} a_0(t). \tag{28}$$

Since $A_0 = 1$, we therefore need that $\lim_{t\to 0} a_0(t) = 1$. This is fulfilled only if $t_0 = 1$.
In addition, we also have to check that

$$\frac{1}{2} = \lim_{\tau\to\infty} C_0(\tau) = \lim_{t\to 0} c_0(t) = \lim_{t\to 0} \frac{1}{1 + a_0(t)}, \tag{29}$$

which is true since $\lim_{t\to 0} a_0(t) = 1$.
This finally leads to the uniform solutions

$$a_0^u(t) = A_0\left(\frac{t}{\varepsilon}\right) + a_0(t) - \lim_{\tau\to\infty} A_0(\tau) = a_0(t), \\ c_0^u(t) = C_0\left(\frac{t}{\varepsilon}\right) + c_0(t) - \lim_{\tau\to\infty} c_0(\tau) = \frac{1}{2}\left(1 + e^{-2t/\varepsilon}\right) + \frac{1}{1 + a_0(t)} - \frac{1}{2} = \frac{e^{-2t/\varepsilon}}{2} + \frac{1}{1 + a_0(t)}. \tag{30}$$

3 Problem

Explain that in order to determine the stability of an equilibrium point u_0 for the model $du/dt = f(u)$, it is usually sufficient to consider the sign of $f(u)$ around u_0. Apply this for making a sketch of the bifurcation diagram indicating stable and unstable equilibrium points in the (μ, u)-plane for the model

$$\frac{du}{dt} = \left(\mu + u^2 - 2u - 1\right)(\mu + u). \tag{31}$$

(It is not necessary to state the exact expressions for the equilibrium points or consider points requiring higher order analysis).

Solution: We observe that $u(t)$ increases when $f(u) > 0$ and decreases when $f(u) < 0$. If we start from a point u_1 near u_0, we will drift towards u_0 if $f(u_1) < 0$ and $u_1 > u_0$, or $f(u_1) > 0$ when $u_1 < u_0$. The opposite gives a drift away from u_0 (Needs to be more careful if $f'(u_0) = 0$).

Relating this to the derivative test requires *Taylor's Formula*:

$$f(u) - f(u_0) = f(u) = \frac{df}{du}(u_0)(u - u_0) + o(|u - u_0|). \tag{32}$$

(not needed for the exam).

Here the equilibrium points occur for

$$\left(\mu + u^2 - 2u - 1\right)(\mu + u) = 0, \tag{33}$$

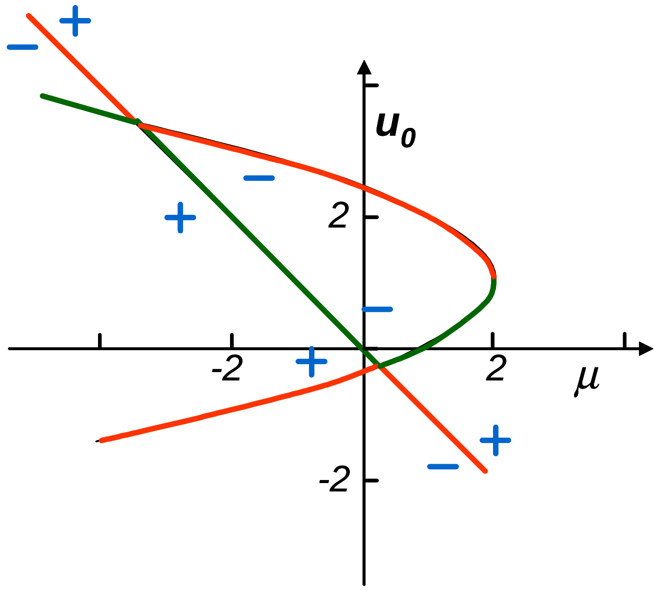

Figure 1: Bifurcation diagram for Problem 3. Red is unstable and green is stable.

that is,

$$u_0 = -\mu, \tag{34}$$

or

$$u_0^2 - 2u_0 - 1 = -\mu. \tag{35}$$

The first equation represents a line and the second a parabola in the (μ, u)-plane, as sketched on Fig. 1. It is convenient first to draw the curves and determine the (unique!) sign of $f(u)$ in each region (shown with blue symbols on the graph). Then stability/instability is determined by moving vertically around the equilibrium points.

4 Problem

A part of a water cleaning system is modelled as a tube (along the x^*-axis) of length L where polluted water flows with constant velocity V. The tube also contains absorbers that remove the pollution. The concentration of pollutant in the water is c^*, measured as amount per length unit of pipe. Similarly, the amount of absorbed pollutant per length unit of pipe is denoted ρ^*. Some of the absorbed pollutant will over time re-enter the water stream. The absorption and re-entering is modelled by the equation

$$\frac{\partial \rho^*}{\partial t^*} = k_1 (A - \rho^*) c^* - k_2 \rho^*. \tag{36}$$

(a)

State the integral conservation law for the pollutant and show that it leads to the differential form

$$\frac{\partial}{\partial t^*}(c^* + \rho^*) + \frac{\partial}{\partial x^*}(V c^*) = 0. \tag{37}$$

Based on the integral law, establish that a discontinuity in the concentrations, moving with velocity U^*, has to fulfil

$$U^* = \frac{c_2^* - c_1^*}{(c_2^* + \rho_2^*) - (c_1^* + \rho_1^*)} V, \tag{38}$$

where (c_1^*, ρ_1^*) and (c_2^*, ρ_2^*) are the concentrations on the respective sides of the discontinuity.

(b)

Introduce suitable scales and show that the equations may be written

$$\frac{\partial}{\partial t}(c+\rho) + \frac{\partial c}{\partial x} = 0, \tag{39}$$

$$\varepsilon \frac{\partial \rho}{\partial t} = (1-\rho)\, c - \beta \rho. \tag{40}$$

Explain the meaning of ε and β (Hint: Use the same scale for ρ^* and c^*).

Assume that the tube is infinitely long in both directions and consider analytic solutions of Eqns. 39 and 40 in the form of "fronts", travelling with velocity U:

$$c(x,t) = C(x - Ut), \tag{41}$$

$$\rho(x,t) = R(x - Ut). \tag{42}$$

We limit ourselves to the special case where $C(\eta)$ and $R(\eta)$ ($\eta = x - Ut$) satisfy

$$\lim_{\eta \to -\infty} C(\eta) = 1, \tag{43}$$

$$\lim_{\eta \to \infty} C(\eta) = 0, \tag{44}$$

$$\lim_{\eta \to -\infty} R(\eta) = \frac{1}{1+\beta}, \tag{45}$$

$$\lim_{\eta \to \infty} R(\eta) = 0. \tag{46}$$

(c)

Insert 41 and 42 into Eqn. 39, integrate once, and use the behaviour at $-\infty$ and ∞ to determine U and a simple relation between C and R. Use this information and Eqn. 40 to determine $C(\eta)$ and $R(\eta)$. How is the behaviour of the solution when $\varepsilon \to 0$?

(Hint: The equation

$$\frac{dy}{d\zeta} = y\left(-1 + \frac{y}{M}\right) \tag{47}$$

has a solution

$$y(\zeta) = M \frac{1}{1 + \exp \zeta} \tag{48}$$

for $0 < y < M$).

(d)

Assume $\varepsilon = 0$ in Eqn. 40 so that the system (39 and 40) simplifies to

$$\rho = \frac{c}{c+\beta}, \ \frac{\partial}{\partial t} Q(c) + \frac{\partial c}{\partial x} = 0, \tag{49}$$

$$Q(c) = c + \frac{c}{c+\beta}, \ -\infty < x < \infty, \ t \geq 0. \tag{50}$$

Consider the initial condition

$$c(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{51}$$

Show that the corresponding solution of Eqn. 49 develops a shock. Determine the shock velocity from the expression in point (a) and compare to the result in (c).

Solution:

(a) Consider a control volume between x_1^* and x_2^* including both the fluid and the absorbers. Since the flux is Vc^*, we obtain

$$\frac{d}{dt^*} \int_{x_1^*}^{x_2^*} (c^* + \rho^*) \, dx^* + Vc^*(x_2^*, t^*) - Vc^*(x_1^*, t^*) = 0. \tag{52}$$

The standard argument, interchanging derivative and integral and the integral mean value theorem leads to

$$\frac{\partial}{\partial t^*}(c^* + \rho^*) + \frac{\partial}{\partial x^*}(Vc^*) = 0. \tag{53}$$

In order to derive U, we put the control volume around the discontinuity. The conservation of pollution requires ('1' refers to the left of the shock, and '2' to the right):

$$\begin{aligned} &\frac{d}{dt^*} \int_{x_1^*}^{x_2^*} (c^* + \rho^*) \, dx^* + Vc^*(x_2^*, t^*) - Vc^*(x_1^*, t^*) \\ &= -\lim_{\Delta t^* \to 0} \frac{[(c_2^* + \rho_2^*) - (c_1^* + \rho_1^*)] U \Delta t^*}{\Delta t^*} + (Vc_2^*) - (Vc_1^*) = 0. \end{aligned}$$

Thus,

$$U^* = \frac{c_2^* - c_1^*}{(c_2^* + \rho_2^*) - (c_1^* + \rho_1^*)} V.$$

(b) The equations and the hint suggest the following scaling:

$$\begin{aligned} x^* &= Lx, \\ t^* &= \frac{L}{V} t \\ \rho^* &= A\rho, \\ c^* &= Ac, \end{aligned} \tag{54}$$

Thus

$$\frac{\partial}{\partial t}(c + \rho) + \frac{\partial c}{\partial x} = 0, \tag{55}$$

$$\varepsilon \frac{\partial \rho}{\partial t} = (1 - \rho) c - \beta \rho, \tag{56}$$

where

$$\varepsilon = \frac{1/(k_1 A)}{L/V}, \tag{57}$$

$$\beta = \frac{1/(k_1 A)}{1/k_2}. \tag{58}$$

Similarly to Problem 2, both ε and β are quotients between time scales. The time scale L/V is the transverse time for the fluid, $1/(k_1 A)$ is the scale for absorption, and $1/k_2$ the typical time for re-entrance of pollution into the stream.

(c) We introduce the trial solution in Eqn. 55 and let $\eta = x - Ut$:

$$-U\frac{dC}{d\eta} - U\frac{dR}{d\eta} + \frac{dC}{d\eta} = 0. \tag{59}$$

Thus,

$$-U\left[C(\eta) + R(\eta)\right] + C(\eta) = \text{const.} \tag{60}$$

Letting $\eta \to \infty$ shows that the constant is equal to 0, whereas $\eta \to -\infty$ gives

$$-U\left(1 + \frac{1}{1+\beta}\right) + 1 = 0, \tag{61}$$

or

$$U = \left(1 + \frac{1}{1+\beta}\right)^{-1}. \tag{62}$$

Since $C(1-U) - UR = 0$, we have

$$C = \frac{U}{1-U}R = (1+\beta)R, \tag{63}$$

and the ratio between C and R is thus constant for all η-s!

Inserting this into the second equation, leads to

$$\begin{aligned} -\varepsilon U\frac{dR}{d\eta} &= (1-R)(1+\beta)R - \beta R \\ &= R - (1+\beta)R^2. \end{aligned} \tag{64}$$

From the equation in the hint, which is a simple modification of the logistic equation, we obtain (not bothering about an arbitrary shift of the origin):

$$\begin{aligned} R(x,t) &= \frac{1}{1+\beta}\frac{1}{1+\exp\frac{x-Ut}{\varepsilon U}}, \\ C(r,t) &= (1+\beta)R(x,t) = \frac{1}{1+\exp\frac{x-Ut}{\varepsilon U}}, \\ U &= \left(1+\frac{1}{1+\beta}\right)^{-1}. \end{aligned} \tag{65}$$

When $\varepsilon \to 0$, the solution approaches a shock occurring at $x = Ut$.

(d) Even if it is simple to work directly with Eqn. 49, the equation may be transformed to our familiar form by first observing that

$$\frac{\partial}{\partial t} Q(c) + \frac{\partial c}{\partial x} = q(c)\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = 0. \tag{66}$$

And since

$$q(c) = \frac{dQ(c)}{dc} = 1 + \frac{\beta}{(c+\beta)^2}, \tag{67}$$

is always strictly positive, we may just as well consider

$$\frac{\partial c}{\partial t} + \frac{1}{q(c)}\frac{\partial c}{\partial x} = 0. \tag{68}$$

The kinematic velocity, $1/q(c)$, is strictly increasing with c, and hence characteristic lines starting from $x < 0$ will overtake the lines starting at $x > 0$, thus leading to a shock situation (the characteristics move into the shock). The speed of the shock was considered in point (a), and the scaled version for the shock velocity is

$$U = \frac{c_2 - c_1}{(c_2 + \rho_2) - (c_1 + \rho_1)} = \frac{0 - 1}{0 - \left(1 + \frac{1}{1+\beta}\right)} = \left(1 + \frac{1}{1+\beta}\right)^{-1}. \tag{69}$$

This is equal to the velocity in (c), and shows that for this case, the continuous front solutions of the full system decay nicely into the shock solution of the degenerate system ($\varepsilon = 0$).