TMA 4195 Mathematical Modeling
December 2001
Solution with additional comments

1 Problem

(a) The number of individuals in a population, N^*, is often modeled in terms of a logistic model,

$$\frac{1}{N^*(t^*)}\frac{dN^*(t^*)}{dt^*} = r\left(1 - \frac{N^*(t^*)}{N_m}\right),\ 0 < r,\ 0 < N_m. \tag{1}$$

Scale the equation and show how to investigate the stability of the equilibrium points.

In closed laboratory studies of bacteria it has turned out to be difficult to find populations following Eqn. 1. Instead, the population, after a while, tends to fall to 0 due to self-poisoning. It is probable this also applies to mankind. The contamination may be due to PCB, long living radioactive waste and hormones influencing the fertility. The following alternative model is therefore proposed for the population of the earth:

$$\frac{1}{N^*(t^*)}\frac{dN^*(t^*)}{dt^*} = r\left(1 - \frac{N^*(t^*)}{N_m}\right) - c\int_{-\infty}^{t^*} N^*(s^*)\,ds^*,\ c > 0. \tag{2}$$

(b) How can we argue for the last term? Show that for this model, the only possible limit when t^* tends to infinity is $N^* = 0$.

(c) Assume that N_m is so large that N^* never approaches N_m, scale the equation and show that we may approximately write

$$\frac{1}{N}\frac{dN}{dt} = 1 - \alpha\int_{-\infty}^{t} N(s)\,ds. \tag{3}$$

Solve Eqn. 3 by introducing $P(t) = \alpha\int_{-\infty}^{t} N(s)\,ds$, and determine how the population and the pollution develop over time.

Hint: The equation $y'' = y'(1-y)$ has, when $\lim_{x\to-\infty} y(x) = 0$, the general solution

$$y(x) = 2\frac{1}{1+e^{-(x-x_0)}} = \left(1 + \tanh\left(\frac{x-x_0}{2}\right)\right),$$

and

$$\frac{d}{dx}\tanh(x) = \frac{1}{\cosh^2 x} = \frac{4}{(e^x + e^{-x})^2}.$$

Solution:

(a) The left-hand side of the equation is the *relative growth rate.* Whenever $N^* << N_m$, we have approximately exponential growth,

$$\frac{dN^*(t^*)}{dt^*} = N^*(t^*)\, r,$$

with solution $N^*(t^*) \propto \exp(rt^*)$. The growth rate decreases as N^* increases, and is 0 when $N^*(t^*) = N_m$, called the *sustainable capacity.* The scales, $N^* = NN_m$ and $t^* = t/r$, lead to

$$\dot{N} = N(1-N) = f(N).$$

The equilibrium solutions are $N_0 = 0$ and $N_1 = 1$. An equilibrium point is stable if $df(N)/dN < 0$, and unstable if $df(N)/dN > 0$ (this can be checked immediately by inserting $N(t) = N_i + n(t)$, $i = 0, 1$, expand and solve for $n(t)$). Here N_0 is unstable, whereas N_1 is stable.

(b) It is reasonable that the contamination per unit of time is proportional to the number of individuals. Thus, the total amount of contamination at time t_0 is proportional to

$$\int_{-\infty}^{t^*} N^*(s^*)\, ds^*,$$

where we have assumed that the contamination stays in the environment without breaking down. Moreover, we assume that the negative influence on the growth rate is proportional to total amount of contamination.

If $\lim_{t^*\to\infty} N^*(t^*) = a > 0$, the RHS of the equation will behave as

$$r\left(1 - \frac{N^*(t^*)}{N_m}\right) - c\int_{-\infty}^{t^*} N^*(s^*)\, ds^* \approx r\left(1 - \frac{a}{N_m}\right) - c\int_{-\infty}^{t^*} a ds^* \underset{t^*\to\infty}{\longrightarrow} -\infty,$$

which is inconsistent with $N^*(t^*) \to a$.

Digression: One simple alternative model, which does not have such dramatic consequences, would be

$$\frac{1}{N^*(t^*)}\frac{dN^*(t^*)}{dt^*} = r\left(1 - \frac{N^*(t^*)}{N_m}\right) - c\int_{-\infty}^{t^*} e^{-(t^*-s^*)/\tau} N^*(s^*)\, ds^*. \tag{4}$$

Here the contamination (or the influence on the growth rate) decreases exponentially with time constant τ, and this allows a certain population to survive. Prove that by introducing

$$S^*(t^*) = \int_{-\infty}^{t^*} e^{-(t^*-s^*)/\tau} N^*(s^*)\, ds^*$$

and check equilibrium points for N^* and S^*.

(c) We scale time with $1/r$. There is no obvious scale for N^*, but we have to use something and in this simple case, what we use does not really matter. It is not reasonable to use N_m since we have assumed that N^* never gets close to N_m, but this does not affect the actual solution. If we set $N^* = NN_m$ and, as suggested, ignore the term $-rN^*/N_m$, we obtain with $N_m N(s) = N^*(s/r)$,

$$\frac{1}{NN_m}\frac{d(NN_m)}{d(t/r)} = r - c\int_{-\infty}^{t^*} N^*(s^*)\,ds^* \tag{5}$$

$$= r - c\int_{-\infty}^{t} N_m N(s)\,\frac{ds}{r}, \tag{6}$$

i.e.,

$$\frac{1}{N}\frac{dN}{dt} = 1 - \alpha\int_{-\infty}^{t} N(s)\,ds, \alpha = cN_m/r^2. \tag{7}$$

In order to solve the equation, let us note that $P' = \alpha N$ and $P'' = \alpha N'$. If this is inserted into the equation 7, we obtain

$$P'' = P'(1-P). \tag{8}$$

Since we obviously have that $\lim_{t\to-\infty} P(t) = 0$, we may apply the given solution:

$$P(t) = 1 + \tanh\left(\frac{t-t_0}{2}\right).$$

(which is the solution of a logistic equation obtained by integrating Eqn. 8 once and using that $P(t) \to 0$ whenever $t \to -\infty$). For N we find

$$N(t) = \frac{1}{\alpha}P'(t) = \frac{1}{2\alpha}\cosh^{-2}\left(\frac{t-t_0}{2}\right).$$

The population reaches its biggest size, $1/2\alpha$, for $t = t_0$. The solution has been drawn for $\alpha = 1$ in fig. 1. The prospects are not particularly good.

Digression: If we observe the population's size and determine when $N''(t_s) = 0$, we can, using $N(t_s)$ and $N'(t_s)$, actually estimate when the population reaches its maximum and its actual size.

The earth's population seems to have passed the turning point $N''(t_s) = 0$ in 1996, when we had, relying on my internet sources,

$$N(t_s) = 5.87 \times 10^9,$$

$$\frac{dN}{dt}(t_s) = 8.3 \times 10^7 \mathrm{y}^{-1}.$$

This will give a maximum, $N_{\max} = 8.8 \times 10^9$ around 2050.

2 Problem

A long thin cylinder of porous sandstone with constant cross section A is placed along the x-axis. The pores constitute a constant fraction Φ of the volume $(0 < \Phi < 1)$ and is filled with water. We assume that the water has constant density and measure the amount of

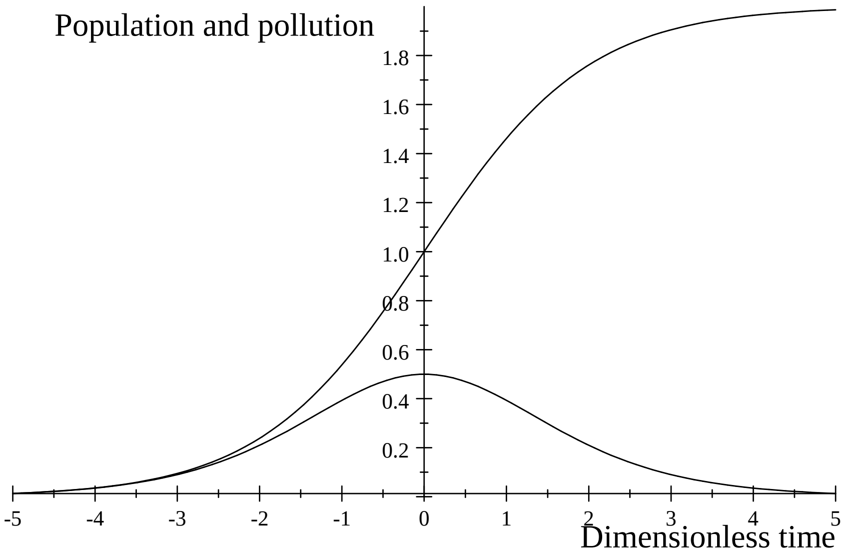

Figure 1: Development of dimensionless population (proportional to $\cosh^{-2}(t/2)$) and pollution (proportional to $1 + \tanh(t/2)$).

water by its volume. The sides of the cylinder are closed, but by applying pressure on the fluid at the input end, it is possible to move water through the stone.

In order to find an expression for the flux of water in the x-direction , j [m^3/(m^{2}s)], we assume it only depends of the water viscosity, μ [kg/ (ms)], the permeability (inverse flow resistance) of the stone, K, [m^2]), and the pressure gradient, $\partial p/\partial x$.

(a) Show that dimension analysis gives

$$j = -k\frac{K}{\mu}\frac{\partial p}{\partial x},$$

where k is a dimensionless constant.

Assume that the cylinder in addition to water also contains oil. All pores are either filled with water or oil, and a volume V of the stone contains a volume $S_o \Phi V$ oil and $S_v \Phi V$ water, where $S_o + S_v = 1$. We assume that water and oil have the same pressure and that the fluxes may be written as

$$j_i = -k_i(S_i)\frac{K}{\mu_i}\frac{\partial p}{\partial x}, \; i = o, v.$$

(b) State the conservation laws for oil and water for a part of the cylinder between $x = a$ and $x = b$. Show that if we apply a pressure gradient such that

$$q = j_o + j_v = \text{constant},$$

we have, for $S \equiv S_v$, the following hyperbolic equation:

$$\Phi \frac{\partial S}{\partial t} + \frac{\partial}{\partial x} f(S) = 0,$$
$$f(S) = \frac{q k_v(S)/\mu_v}{k_o(1-S)/\mu_o + k_v(S)/\mu_v}. \tag{9}$$

(c) Assume that $\mu_o = \mu_v$, $k_o(1-S) = 1 - S^2$ and $k_v(S) = S^2$. Solve the equation (9) for $t > 0$ for a cylinder of length L when

$$S(x, 0) = 1 - x/L, \; 0 \leq x \leq L,$$
$$S(0, t) = 1, \; 0 \leq t.$$

Solution:

(a) We establish the dimension matrix after having observed that pressure is force per unit of surface, and hence

$$\left[\frac{\partial p}{\partial x}\right] = \frac{[p]}{\mathrm{m}} = \frac{\mathrm{kgm}}{\mathrm{s}^2}\frac{1}{\mathrm{m}^3} = \frac{\mathrm{kg}}{\mathrm{s}^2\mathrm{m}^2}.$$

The dimensional matrix is therefore

	j	K	μ	$\partial p/\partial x$
kg	0	0	1	1
m	1	2	-1	-2
s	-1	0	-1	-2

Since the rank is 3, we have only one dimensionless combination. It is easy to see that the given expression matches this. It is also physically reasonable that the flux points in the negative direction of the pressure gradient.

Digression: An equivalent relation holds true for three dimensional flow as well:

$$\mathbf{j} = -k\frac{K}{\mu}\nabla p,$$

However, for inhomogeneous rocks like limestone, K will be a second order tensor. In that case, the water and the oil prefer to flow along the natural layers in the rock, and $\mathbf{j}$ will not necessarily be aligned with $-\nabla p$.

(b) The cylinder lies along the x-axis and has a constant cross-section A. The conservation law for the flow between $x = a$ and $x = b$ is

$$\frac{d}{dt}\int_a^b (S_i\Phi) A dx + j_i(b) A - j_i(a) A = 0, \; i = o, v.$$

The differential formulation can be derived in the usual way by letting $a \to b$:

$$\Phi \frac{\partial S_i}{\partial t} + \frac{\partial j_i}{\partial x} = 0, \; i = o, v.$$

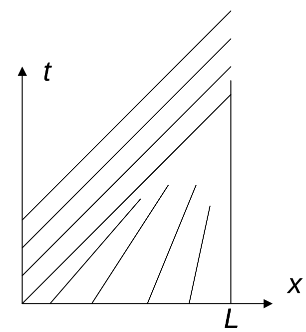

Figure 2: Sketch of the characteristics.

We can eliminate $\frac{\partial p}{\partial x}$ by considering the equations for the flux (the sum should be constant and equal to q):

$$-\left(k_o\left(S_o\right)\frac{K}{\mu_o}+k_v\left(S_v\right)\frac{K}{\mu_v}\right)\frac{\partial p}{\partial x}=q,$$

i.e.

$$\frac{\partial p}{\partial x}=-\frac{q}{\left(k_o\left(S_o\right)\frac{K}{\mu_o}+k_v\left(S_v\right)\frac{K}{\mu_v}\right)}.$$

By putting this into the equation for $S\equiv S_v$, we obtain the given expression,

$$\begin{aligned}\Phi\frac{\partial S}{\partial t}+\frac{\partial j_v}{\partial x}&=\Phi\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}\left(k_v\left(S\right)\frac{K}{\mu_v}\frac{q}{\left(k_o\left(1-S\right)\frac{K}{\mu_o}+k_v\left(S\right)\frac{K}{\mu_v}\right)}\right)\\&=\Phi\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}f\left(S\right)=0.\end{aligned}$$

A more elegant solution would be to say

$$j_v=\frac{qj_v}{q}=\frac{qj_v}{j_0+j_v}=qk_v\left(S\right)\frac{\frac{K}{\mu_v}}{\left(k_o\left(1-S\right)\frac{K}{\mu_o}+k_v\left(S\right)\frac{K}{\mu_v}\right)}.$$

(c) From the given quantities, we see that the flux for water becomes $j_v=\ f\left(S\right)=qS^2$, and the equation reduces to

$$\frac{\partial S}{\partial t}+\frac{q}{\Phi}\frac{\partial\left(S^2\right)}{\partial x}=0.$$

This is a first order hyperbolic equation with kinematic velocity

$$c(S)=\frac{2q}{\Phi}S.$$

The characteristics are sketched in fig. 2.

To study this in more details, we write the equation for a characteristic starting in the point $x_0,\ 0 < x_0 < L$:

$$\begin{aligned} x &= x_0 + \frac{2q}{\Phi} S(x_0) t \\ &= x_0 + \frac{2q}{\Phi}\left(1 - \frac{x_0}{L}\right) t \end{aligned}$$

We now observe that all characteristics meet in the point $x = L,\ t = \frac{\Phi L}{2q}$. Moreover, from the sketch, we see that the solution is $S = 1$ above the characteristic starting at the origin, i.e. $x = \frac{2qt}{\Phi}$. For a point (x_s, t_s) *below* this characteristic, we have

$$x_s = x_0 + \frac{2q}{\Phi}\left(1 - \frac{x_0}{L}\right) t_s,$$

i.e.

$$x_0 = L \frac{2qt_s - x_s\Phi}{2qt_s - \Phi L},$$

and

$$S(x_s, t_s) = S(x_0, 0) = 1 - \frac{x_0}{L} = 1 - \frac{2qt_s - x_s\Phi}{2qt_s - \Phi L} = \frac{x_s - L}{2qt_s/\Phi - L}.$$

3 Problem

(a) The most common model for the traffic of cars along a road leads to a dimensionless flux of cars of the form $j = \rho(1-\rho)$. *Describe how this model is established. State the hyperbolic equation the model leads to (where no cars enter or leave the road). When will the solution develop shocks?*

Consider the model in (a). Between $x = 0$ and $x = 1$ there is a reduction in the speed limit so that the maximal speed reduces to $1/2$, while the maximum density remains the same. We assume that a linear relation between the car velocity and the density also applies for this part.

(b) Which condition on the flux of cars has to hold at $x = 0$ *and* $x = 1$*? Find the solution* $\rho(x,t)$ *for* $t > 0$ *and all* x *when*

$$\rho(x,0) = \begin{cases} 1/2, & x < 1, \\ 0, & x > 1. \end{cases}$$

(Hint: The density ρ between 0 and 1 remains constant for all $t \geq 0$).

Solution:

(a) The model starts by defining a density ρ^* which lies between 0 and a maximal density $\rho_{\max}$. The cars' speed is assumed to be a linear function of ρ^*, $v^* = v_{\max}\left(1 - \frac{\rho^*}{\rho_{\max}}\right)$. Thus, the flux becomes $j^* = \rho^* v^* = \rho^* v_{\max}\left(1 - \frac{\rho^*}{\rho_{\max}}\right)$. The scaling

$$\begin{aligned} \rho^* &= \rho\rho_{\max}, \\ x^* &= Lx, \\ t^* &= (L/v_{\max})t, \end{aligned}$$

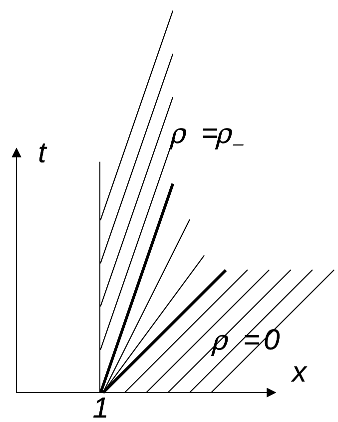

Figure 3: Characteristics for the solution in the region $x > 1$.

gives $j = \rho(1-\rho)$, and the equation

$$\rho_t + c(\rho)\,\rho_x = 0,$$
$$c(\rho) = \frac{dj}{d\rho} = 1 - 2\rho.$$

Since $c(\rho)$ decreases when ρ increases, a situation where $\rho(x_1, t) < \rho(x_2, t)$ will, for $x_1 < x_2$, develop a shock.

(b) In $x = 0$ and $x = 1$ there are no possibilities for cars to accumulate. Thus *the flux has to be continuous in these points for $t > 0$.*

Between $x = 0$ and 1, the flux will have the form

$$\frac{1}{2}\rho(1-\rho),$$

Using the hint, we assume that $\rho = 1/2$ so that the flux is $1/8$ in this region *for all $t \geq 0$.* Then, it remains to find the solution for $x < 0$ and $x > 1$. In these regions, $j = 1/8$, corresponds to 2 possible densities, namely the solutions of

$$\frac{1}{8} = \rho(1-\rho).$$

i.e.

$$\rho_+ = \frac{1}{2} + \frac{1}{4}\sqrt{2},$$
$$\rho_- = \frac{1}{2} - \frac{1}{4}\sqrt{2}.$$

The situation in the region $x > 1$ is sketched in fig. 3.

"Limit-characteristics" which starts in $x = 1$ have equations

$$x = 1 + t,$$
$$x = 1 + (1 - 2\rho_-)t = 1 + \frac{1}{2}\sqrt{2}t.$$

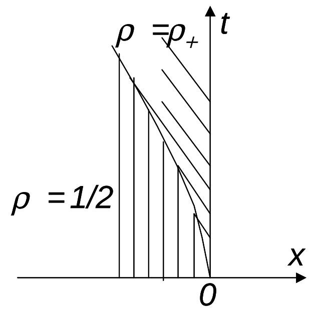

Figure 4: Sketch of the solutions to the left of $x = 0$.

In the region between these characteristics, we have a rarefaction wave such that

$$x = 1 + (1 - 2\rho)t,$$

i.e.

$$\rho(x,t) = \frac{1 - x + t}{2t}$$

For $x < 0$ we have a situation where $\rho = 1/2$ corresponds to a flux $\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$. It is *impossible* to have such a flux immediately to the left side of $x = 0$, since the flux immediately to the right of $x = 0$ can be at most $1/8$. What is happening is that a queue is forming to the left of $x = 0$ with density $\rho = \rho_+$. In the back side of this queue we get a shock. The situation is sketched in fig. 4.

We can deduce the shock's track from the differential equation

$$\frac{dx}{dt} = U = \frac{1/8 - 1/4}{\rho_+ - 1/2} = \frac{1/8 - 1/4}{\frac{1}{2} + \frac{1}{4}\sqrt{2} - 1/2} = -\frac{\sqrt{2}}{4}.$$

In other words, the shock has a *constant speed* and is governed by the equation

$$x = -\frac{\sqrt{2}}{4}t$$

which expresses a straight line starting in the origin.

The solution may be summed up as follows:

		ρ
$x < 0$	$x < -\frac{\sqrt{2}}{4}t$	$\frac{1}{2}$
$x < 0$	$x > -\frac{\sqrt{2}}{4}t$	$\frac{1}{2} + \frac{1}{4}\sqrt{2}$
$0 < x < 1$	$t \geq 0$	$\frac{1}{2}$
$1 < x$	$t < x - 1$	0
$1 < x$	$x - 1 < t < (x - 1)\sqrt{2}$	$\frac{1-x+t}{2t}$
$1 < x$	$(x - 1)\sqrt{2} < t$	$\frac{1}{2} - \frac{1}{4}\sqrt{2}$

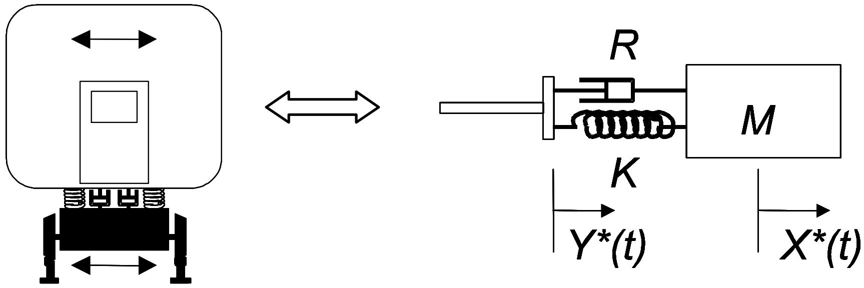

Figure 5: A spring-dashpot equivalent for the transmission between the bogie and the car.

4 Problem

In connection with the breakdown of the bogie shafts on the Signatur trains, the manufacturing company Adtrans carried out measurements of the rails between Oslo and Kristiansand. The measurement of the horizontal deviations is a stochastic process, $Z(x)$, where x is the position along the rail and Z is the deviation. We choose the zero point such that the expectation, $\mathbb{E}(Z(x)) = 0$, and assume that $Z(x)$ is weakly stationary in x. The spectrum of Z has the form

$$S_Z(\kappa) = \begin{cases} A, & 0 < \kappa_1 \leq |\kappa| \leq \kappa_2, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

(In order to differ between space and time, we use κ instead of ω when we have a process dependent on x).

(a) Show that the deviation we observe when we sit on a train that is moving with constant speed U,

$$Y(t) = Z(x_0 + Ut),$$

is a weakly stationary process in time. Determine the covariance function and show that the spectrum of $Y(t)$ can be expressed as

$$S_Y(\omega) = \frac{1}{U} S_Z\left(\frac{\omega}{U}\right).$$

Determine the variance of Y and sketch how $S_Y(\omega)$ looks for large and small velocities.

We shall represent the bogie and the wagon by the simple mechanical model shown on Fig. 5. The piston (bogie) to the left is attached to the wagon with mass M by means of a linear damper and a spring. The motion $Y^*(t)$ of the piston induces a motion $X^*(t)$ of the mass.

(b) Show that the motion of the wagon, $X(t)$, after a suitable scaling, can be written

$$\frac{d^2X}{dt^2} + \delta \frac{dX}{dt} + X(t) = \delta \frac{dY}{dt} + Y(t). \tag{11}$$

Express the spectrum of X by means of the spectrum of Y.

(c) For which constant δ will the variance of X have its minimum when the dimensionless spectrum of Y is constant from $\omega = 0$ up to a dimensionless frequency $\omega_m \gg 1$?

(d) The force (or better, the torque) on the shafts may be expressed from Eqn. 11 as

$$F(t) = r\left(\delta\left(\frac{dX}{dt} - \frac{dY}{dt}\right) + X(t) - Y(t)\right).$$

Show, by a rough estimate, that the variance of F is proportional to U^2 if the spectrum is the same as in (a).

Formulae for Problem 4:

$$\hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt,\ f(t) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{f}(\omega)\,d\omega.$$

$$Y(t) = h * X(t) = \int_{-\infty}^{\infty} h(t-\tau)X(\tau)\,d\tau \implies S_Y(\omega) = (2\pi)^2\left|\hat{h}(\omega)\right|^2 S_X(\omega).$$

$$S_{aX+bdX/dt+cd^2X/dt^2}(\omega) = \left|a + bi\omega - c\omega^2\right|^2 S_X(\omega).$$

$$\int_0^{\infty}\left|\frac{1+i\delta x}{1+i\delta x - x^2}\right|^2 dx = \frac{\pi}{2}\frac{\delta^2+1}{\delta}.$$

Solution:

The idea for this problem comes from a real study which *Adtrans* carried out at *ECMI Modeling Week* in Lund, Sweden in the summer 2000.

(a) We see that the average value of Y is 0 for all t,

$$\mathbb{E}(Y(t)) = \mathbb{E}(Z(x_0 + Ut)) = 0.$$

Furthermore, we have

$$\begin{aligned}\mathrm{Cov}(Y(t_1), Y(t_2)) &= \\ &= \mathrm{Cov}(Z(x_0+Ut_1), Z(x_0+Ut_2)) \\ &= C_Z(U(t_2 - t_1)),\end{aligned}$$

since Z was weakly stationary. But this is just what is needed for Y to be weakly stationary:

$$C_Y(t) = \mathrm{Cov}(Y(t_0), Y(t_0+t)) = C_Z(Ut).$$

For the spectrum we obtain

$$\begin{aligned}
S_Y(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} C_Y(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} C_Z(Ut)dt \\
&= \frac{1}{2\pi}\frac{1}{U}\int_{-\infty}^{\infty} e^{-i(\omega/U)Ut} C_Z(Ut)d(Ut) \\
&= \frac{1}{U}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega/U)s} C_Z(s)ds \\
&= \frac{1}{U} S_Z\left(\frac{\omega}{U}\right).
\end{aligned}$$

We note that the variance Y becomes

$$\begin{aligned}
\mathrm{Var}(Y) &= 2\int_0^{\infty} \frac{1}{U} S_Z\left(\frac{\omega}{U}\right) d\omega = 2\int_0^{\infty} S_Z(\kappa)\, d\kappa \\
&= 2\int_{\kappa_1}^{\kappa_2} A d\kappa = 2A(\kappa_2 - \kappa_1).
\end{aligned}$$

In other words, the variance is the same for Z and Y, no matter which speed the train reaches. We leave to the reader to sketch S_Y for different values of U.

(b) The equation follow immediately by Newton's law:

$$M\frac{d^2X^*}{dt^{*2}} = R\left(\frac{dY^*}{dt^*} - \frac{dX^*}{dt^*}\right) + K(Y^* - X^*).$$

We let the scale for X^* be as the scale for Y^*, namely standard deviation of Y^*, σ_{Y^*}. The eigenfrequency of the system (for $R = 0$) is $\omega_0 = \sqrt{K/M}$, and consequently we can use ω_0^{-1} as a time scale. We enter this into the equation and obtain immediately

$$\frac{d^2X}{dt^2} = \frac{R}{K}\omega_0\left(\frac{dY}{dt} - \frac{dX}{dt}\right) + (Y - X),$$

which is identical to the equation in the problem when $\delta = \frac{R}{K}\omega_0$. Transfer function of the derivative operator is $(i\omega)$, and consequently

$$\begin{aligned}
S_{\ddot{X}+\delta\dot{X}+X} &= \left|-\omega^2 + i\delta\omega + 1\right|^2 S_X, \\
S_{\delta\dot{Y}+Y} &= \left|i\delta\omega + 1\right|^2 S_Y.
\end{aligned}$$

Thus,

$$S_X(\omega) = \left|\frac{1 + i\delta\omega}{1 + i\delta\omega - \omega^2}\right|^2 S_Y(\omega). \tag{12}$$

(c) Since we know that $\omega_m >> 1$,

$$\begin{aligned}\operatorname{Var}(X) &= 2\int_0^{\omega_m}\left|\frac{1+i\delta\omega}{1+i\delta\omega-\omega^2}\right|^2 A d\omega \\ &\approx 2A\int_0^{\infty}\left|\frac{1+i\delta\omega}{1+i\delta\omega-\omega^2}\right|^2 d\omega \qquad (13)\\ &= 2A\frac{\pi}{2}\frac{\delta^2+1}{\delta} = A\pi\frac{\delta^2+1}{\delta}. \qquad (14)\end{aligned}$$

The minimum for the right-hand side is reached for $\delta = 1$ since

$$\frac{d}{d\delta}\left(\frac{\delta^2+1}{\delta}\right) = \frac{\delta^2-1}{\delta^2}.$$

Note that as ω_m increases, $\operatorname{Var}(X)$ tends to a constant $A\pi\frac{\delta^2+1}{\delta}$, while the variance of Y is $2A\omega_m$, and goes to infinity!

(d) What matters for the variance when $U \to \infty$ is how the spectrum of F behaves for high frequencies. We know from the form of F that the spectrum of F is a function of ω multiplied by the spectrum of Y. This is different from 0 only between $U\kappa_1$ and $U\kappa_2$. The spectrum of $\delta\dot{Y} + Y$ is therefore

$$|i\delta\omega + 1|^2 S_Y(\omega),$$

which behaves as $\delta^2\omega^2 S_Y(\omega)$ for big ω. Similarly, we obtain for the spectrum of $\delta\dot{X} + X$,

$$|i\delta\omega+1|^2 S_X(\omega) = |i\delta\omega+1|^2\frac{|i\delta\omega+1|^2}{|1+i\delta\omega-\omega^2|^2}S_Y(\omega) \backsim S_Y(\omega).$$

Therefore, it appears that the spectrum of $\delta\dot{Y} + Y$ dominates.

Alternatively, one can, by using the formula for transfer functions, write

$$\begin{aligned}F(t) &= r\left(\delta\left(\frac{dX}{dt}-\frac{dY}{dt}\right)+X(t)-Y(t)\right) = r\left(\left(\delta\frac{d}{dt}+I\right)X-\left(\delta\frac{d}{dt}+I\right)Y\right)\\ &= r\left(\left(\delta\frac{d}{dt}+I\right)HY-\left(\delta\frac{d}{dt}+I\right)Y\right) = r\left(\left(\delta\frac{d}{dt}+I\right)H-\left(\delta\frac{d}{dt}+I\right)\right)Y,\end{aligned}$$

and this gives the exact relation

$$S_F(\omega) = r^2\left|\frac{(i\delta\omega+1)^2}{1+i\delta\omega-\omega^2} - i\delta\omega - 1\right|^2 S_Y(\omega). \qquad (15)$$

For large frequencies,

$$S_F(\omega) \approx (r\delta)^2\omega^2 S_Y(\omega).$$

The simplest solution is however to observe that the force $F(t)$ is equal to $-\ddot{X}(t)$, and consequently,

$$S_F(\omega) = r^2(i\omega)^4 S_X(\omega) = r^2(i\omega)^4\left|\frac{i\delta\omega+1}{1+i\delta\omega-\omega^2}\right|^2 S_Y(\omega), \qquad (16)$$

which is identical to 15.

With the spectrum from (a), this becomes

$$\begin{aligned}
\operatorname{Var} F &\approx (r\delta)^2 \frac{A}{U} \int_{U\kappa_1}^{U\kappa_2} \omega^2 d\omega \\
&= (r\delta)^2 \frac{A}{U} \frac{1}{3} \left((U\kappa_2)^3 - (U\kappa_1)^3 \right) \\
&= (r\delta)^2 \frac{A}{3} U^2 \left(\kappa_2^3 - \kappa_1^3 \right).
\end{aligned}$$

Consequently $\operatorname{Var} F \propto U^2$; which may not be so surprising.