TMA 4195 Mathematical Modeling
December 2002
Solution with additional comments

1 Problem

(a) Which axioms about nature form the basis of dimensional analysis? What is the content of Buckingham's-Pi theorem, and how can you apply the theorem to show that the period of a mathematical pendulum is independent of its mass?

(b) In forestry it is necessary to estimate the volume (v) of trees by measuring the diameter at the base and their height (h) (this may be carried out by triangularization). An example in Minitab suggests the following regression model for American cherry trees:

$$v^{1/3} = \beta_0 + \beta_1 d + \beta_2 h + \beta_3 d^2 \tag{1}$$

(β_i, $i = 0, \cdots, 3$ are regression coefficients). Show that dimensional analysis instead suggests a formula $\pi_1 = \phi(\pi_2)$. Give examples of the form of ϕ for "simple" trees.

Solution:

Buckingham:

- All relations have to be dimensionally correct
- No relation depends on a particular set of units

If there is a relation

$$\Phi(R_1, \cdots, R_N) = 0, \tag{2}$$

there is also an equivalent relation

$$\Psi(\pi_1, \cdots, \pi_{N-r}) = 0, \tag{3}$$

where $\pi_1, \cdots, \pi_{N-r}$ are dimensionless combinations formed by using r core variables with independent dimensions, and where r also is the rank of the dimension matrix.

The period of a mathematical pendulum depends on its length (L), deflection (θ_0), acceleration of gravity (g), and mass (m). To apply dimensional analysis, the mass m has to enter in at least one dimensionless combination with the others. This is not possible since m is the only quantity which contains kg.

(b)

We assume a physical relation

$$\Phi(v, d, h) = 0. \tag{4}$$

Here, just one unit (meter) is involved in all physical quantities. Thus, we have $3 - 1 = 2$ dimensionless combinations π_1 and π_2, and a relation $\pi_1 = \phi(\pi_2)$. We may choose d as a core variable such that $\pi_1 = v/d^3$ and $\pi_2 = h/d$. Then,

$$\frac{v}{d^3} = \phi\left(\frac{h}{d}\right). \tag{5}$$

For a conical "tree" without branches $v = \pi h d^2/12$, and

$$\frac{v}{d^3} = \frac{\pi}{12}\frac{h}{d}, \tag{6}$$

The function ϕ is then linear, $\phi(x) = \frac{\pi x}{12}$.

2 Problem 2

A river flows into an ocean basin. The river brings sand and clay so that the basin is filled up over time. We shall formulate a simple one-dimensional model for how the basin is filled, and we assume that the basin spans from $x = 0$ to $x = +\infty$ and has a constant depth h at $t = 0$. Conditions across (in the y-direction) are constant.

The amount of sand and clay which settle on the bottom per time and area unit is $q(x,t)$. We write the depth $z = b(x,t)$, $x \geq 0$, $t \geq 0$, and let $b(x,t) \leq 0$. If the bottom tilts, the particles on the bottom will continue to move, and it has been found that the mass flux will be proportional to the slope, that is, the volume flux may be written

$$j = -k\frac{\partial b}{\partial x}. \tag{7}$$

(a) Write the conservation equation in integral form for a part of the bottom, $x_0 \leq x \leq x_1$, and show that we for the differential formulation obtain an equation which is identical to the heat equation,

$$\frac{\partial b}{\partial t} = k\frac{\partial^2 b}{\partial x^2} + q. \tag{8}$$

(b) *Assume that all sand and clay enter at $x = 0$ (i.e. $q = 0$ for $x > 0$), and that the amount entering is always sufficient for Eqn. 8 to hold for $t > 0$. Argue that the solution to Eqn. 8 will be a similarity solution in this case, and find $b(x,t)$ for $x \geq 0$ and $t > 0$.*

(Hint: The equation

$$\frac{d^2y}{d\eta^2} + \frac{\eta}{2}\frac{dy}{d\eta} = 0 \tag{9}$$

has the general solution $A + B\,\mathrm{erf}(\eta/2)$, where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-s^2)\,ds$).

(c) *A more realistic scenario is that the shore, $s(t)$, will move forward with time. Assume that a constant volume of sand and clay enters the basin per time unit, q_0, and that all sand and clay enter at the shore.*

The solution will then be stationary with respect to the shore and may be written by means of a function b_0 so that

$$b(x,t) = \begin{cases} 0 & x \leq s(t) = Ut + x_0 \\ b_0(x - Ut - x_0) & x > Ut + x_0 \end{cases} \tag{10}$$

Determine the velocity U and the solution in this case.

Solution:

The background for this problem is taken from the field *Basin Modelling*, i.e. mathematical modelling of the geological processes forming what much later becomes an oil reservoir.

(a) We do the computations with a segment with width B and introduce density, flux and sources. Here we assume that the density of sand is a constant ρ. The flux then becomes ρj since j was supposed to be the *volume* flux. Similarly, the source function will be $\rho q(x,t)$. (However, both B and ρ drop out from the relations at the end, such that we could as well compute per unit width, and with $\rho = 1$).

Our control volume has width B and extends from $x = x_0$ to $x = x_1$. Thus, we obtain the general (one-dimensional) conservation law

$$\frac{d}{dt}\int_{x_0}^{x_1} \rho B\left(b(x,t) - (-h)\right)dx + \left(-k\frac{\partial b}{\partial x}(x_1,t) + k\frac{\partial b}{\partial x}(x_0,t)\right)\rho B = \int_{x_0}^{x_1} q(x,t)(\rho B)\,dx, \tag{11}$$

or

$$\frac{d}{dt}\int_{x_0}^{x_1}(b(x,t)+h)\,dx + \left(-k\frac{\partial b}{\partial x}(x_1,t) + k\frac{\partial b}{\partial x}(x_0,t)\right) = \int_{x_0}^{x_1} q(x,t)\,dx. \tag{12}$$

If we let $x_1 \to x_0$ and divide by $(x_1 - x_0)$, we obtain

$$\frac{\partial}{\partial t}(b+h) = \frac{\partial b}{\partial t} = k\frac{\partial^2 b}{\partial x^2} + q. \tag{13}$$

(b) In this case, the source is localized at $x = 0$, such that the equation for $x > 0$ becomes just $b_t = k b_{xx}$. We scale b with h and write the solution as

$$b = hf(x,t,k). \tag{14}$$

It is not obvious that we have a similarity solution since the depth h could be a length scale, but this length is not associated with the horizontal length. The problem is completely equivalent to a heat conduction problem where the temperature is constant and equal to T_0 at $x = 0$, and T_∞ when $x \to \infty$. The temperature could then be written as $T(x,t) = T_0 + (T_\infty - T_0)\,\tau(x,t,k)$, and we obtain a similarity solution for τ. Similarly to the temperature, we should be able to write the solution for b as

$$b = -h\beta\left(\frac{x}{\sqrt{kt}}\right) = -h\beta(\eta),\ \eta = \frac{x}{\sqrt{kt}}, \tag{15}$$

where $\beta(0) = 0$ and $\beta(\eta) \to 1$ when $\eta \to \infty$. Entering this into the equation after divided by $-h$ leads to

$$\beta_t - k\beta_{xx} = -\frac{1}{2}\frac{x}{\sqrt{k}}\frac{1}{t^{3/2}}\beta' - k\frac{1}{kt}\beta'' = 0, \tag{16}$$

or

$$\beta'' + \frac{\eta}{2}\beta' = 0. \tag{17}$$

This is the equation given in the problem, and we immediately see that

$$b(x,t) = -h\,\mathrm{erf}\left(\frac{x}{\sqrt{kt}}\right). \tag{18}$$

(c) If we assume the given form for b, the amount of sand and clay between $s(t)$ and $x = \infty$ will be constant. This means that the speed at which the shore advances, $ds/dt = U$, has to match q_0, i.e.

$$hB \cdot U = q_0 B, \tag{19}$$

or

$$U = \frac{q_0}{h}. \tag{20}$$

(The argument is simply that the amount per unit of time has to be equal to the shore's growth per unit of time. Note that $[q_0] = \mathrm{m}^3\mathrm{s}^{-1}\mathrm{m}^{-1}$).

If we let $\eta = x - Ut - x_0$ and put in b into the equation where $x > Ut + x$, we obtain

$$-Ub_0' = kb_0'', \tag{21}$$

with general solution

$$b_0(\eta) = A + B\exp\left(-\frac{U}{k}\eta\right). \tag{22}$$

It is required that

$$\begin{aligned} b_0(0) &= 0, \\ b_0(\infty) &= -h, \end{aligned} \tag{23}$$

such that the solution becomes

$$b_0(\eta) = h\left(\exp\left(-\frac{U}{k}\eta\right) - 1\right) \tag{24}$$

Introducing the original variables leads to

$$b(x,t) = \begin{cases} 0, & x \le s(t) = Ut + x_0, \\ h\left(\exp\left(-\frac{U}{k}(x - Ut - x_0)\right) - 1\right), & x > Ut + x_0. \end{cases} \tag{25}$$

3 Problem 3

It has been suggested to model the population of *King Crab*, $K(t)$, in Varangerfjorden by means of the following differential equation:

$$\frac{dK}{dt} = rK\left(1 - \frac{K}{M}\right)\left(\frac{K}{m} - 1\right),\ 0 < m < M. \tag{26}$$

(a) What are the properties of the model and what are the equilibrium populations? Show how we can use linear stability analysis to determine the stability of the equilibrium populations, and make sketches of the development of $K(t)$ for $t > 0$.

(b) A simplified model which also includes fishermen, $F(t)$, has, after scaling, the following form:

$$\begin{aligned} \frac{dF}{dt} &= -\frac{F}{2} + KF, \\ \frac{dK}{dt} &= K(1 - K) - F \end{aligned} \tag{27}$$

What kind of qualitative properties are built into this model? The path $F = 3K(K-1)/2,\ 0 \le K \le 1$, is a solution to Eqn. 27 and divides the first (F, K)-quadrant into two separate regions. What happens when the system is in the unbounded region?

(c) Determine the equilibrium points for the model in Eqn. 27 and determine their type. What happens in the region defined by $0 \le K \le 1$ and $0 \le F \le 3K(K-1)/2$?

Solution:

(a) Let us observe that the population increases if

$$m < K < M. \tag{28}$$

The equilibrium points are given by

$$K = 0, m, M, \tag{29}$$

and by sketching the third degree polynomial $f(K)$ which constitute the right-hand sides, we find

$$\begin{aligned} f'(0) &< 0, \\ f'(m) &> 0, \\ f'(M) &< 0. \end{aligned} \tag{30}$$

We omit the simple argument to decide stability/instability. Here we just note that 0 and M are stable, while m is unstable. A population less than m will die out.

(b) We see that

- if we do not have fishing activity, the growth follows the logistic model
- fishing activities will reduce the population
- if the population $K < 1/2$ is not so interesting to fish and the amount of fishers decreases
- if $K > 1/2$ the amount of fishers increases.

Outside the limit trajectory, F will be larger than $\max(0, 3K(1-K)/2)$, and this means that $\frac{dK}{dt} < 0$. Thus, the population dies out. This argument is not quite rigorous, since one should check what is happening in the neighborhood of singularities which lie on the border of the region.

(c) The singularities are given by

$$\begin{aligned} -\frac{F}{2} + KF &= 0, \\ K(1-K) - F &= 0, \end{aligned} \tag{31}$$

with solutions $\{F = 0, K = 0\}$,$\{F = 0, K = 1\}$ and $\left\{F = \frac{1}{4}, K = \frac{1}{2}\right\}$.

The linearized system for $\{F = 0, K = 0\}$ has form

$$\frac{d}{dt}\begin{bmatrix} f \\ k \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} f \\ k \end{bmatrix} \tag{32}$$

with eigenvalues $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = 1$, hence the point is a *saddle point*. This is also the case for $\{F = 0, K = 1\}$. By linearizing around the point $\left\{F = \frac{1}{4}, K = \frac{1}{2}\right\}$ we find, by introducing $F = \left(\frac{1}{4} + f\right)$ and $K = \left(\frac{1}{2} + k\right)$ that

$$\frac{d}{dt}\begin{bmatrix} f \\ k \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ -1 & 0 \end{bmatrix}\begin{bmatrix} f \\ k \end{bmatrix} \tag{33}$$

This gives $\lambda_1 = -\frac{1}{2}i$ and $\lambda_2 = \frac{1}{2}i$, and consequently, $\left\{F = \frac{1}{4}, K = \frac{1}{2}\right\}$ is a *center* according to the linear analysis. More generally it could be a stable or unstable focus. If the point really is a center, the variations will be periodical, i.e. they will continue indefinitely.

It is possible to show that the point is a center, and the simplest is to introduce a new crab-variable,

$$K = \frac{1}{2} + \kappa \tag{34}$$

The system is then

$$\begin{aligned} \frac{dF}{dt} &= \kappa F, \\ \frac{d\kappa}{dt} &= \frac{1}{4} - \kappa^2 - F \end{aligned}$$

and the trajectories become symmetric with respect to the axis $\kappa = 0$. This can only happen if the trajectories around $\left\{F = \frac{1}{4}, \kappa = 0\right\}$ lying inside the limiting path are closed.

Digression: One can linearize such systems once and for all. If

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{35}$$

is a autonomous system with a differentiable right-hand side, we may Taylor-expand around the singular point $\mathbf{x}_0$ and write

$$f_1(\mathbf{x}_0+\mathbf{y}) = f_1(\mathbf{x}_0) + \sum_{i=1}^{N} \frac{\partial f_1}{\partial x_i}(\mathbf{x}_0)\, y_i + HOT, \tag{36}$$

$$\cdots$$

$$f_N(\mathbf{x}_0+\mathbf{y}) = f_N(\mathbf{x}_0) + \sum_{i=1}^{N} \frac{\partial f_N}{\partial x_i}(\mathbf{x}_0)\, y_i + HOT, \tag{37}$$

The short-hand notation is

$$\begin{aligned} \mathbf{f}(\mathbf{x}_0 + \mathbf{y}) &= \mathbf{f}(\mathbf{x}_0) + \mathbf{A}\mathbf{y} + HOT, \\ \mathbf{A} = \{a_{ij}\},\ a_{ij} &= \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0). \end{aligned} \tag{38}$$

Once plugged into the system, this gives

$$\frac{d\left(\mathbf{x}_0+\mathbf{y}\right)}{dt}=\dot{\mathbf{y}}=\mathbf{f}\left(\mathbf{x}_0\right)+\mathbf{A}\mathbf{y}+HOT=\mathbf{A}\mathbf{y}+HOT, \tag{39}$$

and the linearized system becomes

$$\dot{\mathbf{y}}=\mathbf{A}\mathbf{y}.$$

Here the **A**-matrix is

$$\mathbf{A}=\begin{bmatrix}-\frac{1}{2}+K & F\\ -1 & 1-2K\end{bmatrix}, \tag{40}$$

and it is simple to insert in the singular points and find the eigenvalues.

4 Problem 4

In order to model the crabs' migration along Finnmarkskysten we shall assume they perform a "random walk", so that the flux is $\mathbf{j}^*=-D\nabla^*\rho^*$, where $\rho^*\left(\mathbf{x}^*,t^*\right)$ is the density of crab (amount per area unit), D is a constant, and $\nabla^*=\frac{\partial}{\partial x^*}\hat{\imath}_x+\frac{\partial}{\partial y^*}\hat{\imath}_y$. The growth of the crab population (per area unit) occurs according to a simplified model as in problem 3,

$$q\left(\mathbf{x}^*,t^*\right)=r\rho^*\left(\mathbf{x}^*,t^*\right)\left[1-\frac{\rho^*\left(\mathbf{x}^*,t^*\right)}{\rho_{\max}}\right]. \tag{41}$$

We do not consider fishing in this case.

(a) Formulate the conservation equation for crab on integral form for a region of the ocean and show that the differential formulation leads to a non-linear diffusion equation, which, after scaling, may be written

$$\rho_t=\nabla^2\rho+\rho\left(1-\rho\right). \tag{42}$$

Below we consider Eqn. 42 in one space dimension (x),

$$\rho_t=\rho_{xx}+\rho\left(1-\rho\right). \tag{43}$$

(b) Eqn. 43 has constant solutions $\rho_0=0$ and $\rho_1=1$. Check the stability of these solutions by introducing solutions of the form $\rho\left(x,t\right)=\rho_j+a\left(t\right)\exp(ikx)$, $j=0,1$, $|a|\ll 1$.

(c) It may be shown (but should not be derived here) that the Eqn. 43 has a special solution

$$\rho\left(x,t\right)=\left(1+\exp\left(\sqrt{\frac{1}{6}}x-\frac{5}{6}t\right)\right)^{-2}. \tag{44}$$

What kind of situation does this solution describe? Estimate the time it takes from the crab is observed at a certain location until the population is of the order of the maximum when $r=0.3\text{year}^{-1}$.

Solution:

(a) Let us consider a two-dimensional region R and the standard law,

$$\frac{d}{dt^*}\int_R \rho^* dA + \int_{\partial R} \mathbf{j}^* \cdot \hat{n} dS = \int_R q\left(\mathbf{x}^*, t^*\right) dA. \tag{45}$$

By applying the divergence theorem and moving the derivative side inside the integral, we find

$$\int_R \left(\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot \mathbf{j}^* - q\right) dA. \tag{46}$$

Since this holds for all R,

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot \left(-D\nabla^* \rho^*\right) - r\rho^*\left(\mathbf{x}^*, t^*\right)\left[1 - \frac{\rho^*\left(\mathbf{x}^*, t^*\right)}{\rho_{\max}}\right] = 0. \tag{47}$$

We scale the equation by using

$$\rho^* = \rho_{\max}\rho. \tag{48}$$

For time we ignore diffusion and assume that ρ^* is small. Then, since $\frac{\partial \rho^*}{\partial t} \approx r\rho^*$, the obvious time scale is $T = r^{-1}$. Finally, we decide the length scale X by considering diffusion:

$$D\frac{T}{X^2} = \mathcal{O}(1). \tag{49}$$

In other words,

$$X = \sqrt{DT} = \sqrt{Dr^{-1}}. \tag{50}$$

This gives

$$\frac{\partial \rho_{\max}\rho}{r^{-1}\partial t} + \frac{\rho_{\max}}{Dr^{-1}}\nabla \cdot (-D\nabla\rho) - r\rho_{\max}\rho\left[1 - \rho\right] = 0, \tag{51}$$

which returns the given equation.

This equation is named *Fisher's equation*, after the statistician R. A. Fisher who first used it to model the diffusion of genetic features.

(b)

It is obvious that ρ_0 and ρ_1 are solutions. Linearizing around $\rho = 1$, in other words $\rho = 1 + a(t)\exp(ikx)$, gives

$$\dot{a}\exp(ikx) = -k^2 a\exp(ikx) - a\exp(ikx) + HOT. \tag{52}$$

To the leading order, we have

$$\dot{a} = -\left(k^2 + 1\right) a. \tag{53}$$

Thus, we see that, since the solution is $a(t) \propto \exp\left[-\left(k^2+1\right)t\right]$, this perturbation dies out regardless the size of k. This means that all perturbation $p(x,t)$ where

$$p(x,t) = \int_{-\infty}^{\infty} e^{ikx} a(k,t)\, dk \tag{54}$$

die out.

If we do the same around $\rho = 0$, we end up with

$$\dot{a} = \left(1 - k^2\right) a, \tag{55}$$

Now, a critical situation occurs whenever $k^2 < 1$. Such "long-wave" perturbations will grow in time and $\rho = 0$ is not unconditionally stable.

(c) Let us first note that the flux

$$f(\alpha) = \frac{1}{\left(1 + \exp(\alpha)\right)^2} \tag{56}$$

tends to 1 when $\alpha \to -\infty$ and to 0 when $\alpha \to \infty$. The transition from 1 to 0 happens in the region around $\alpha = 0$:

$$\begin{aligned} f(-5.3) &\approx 0.99, \\ f(2.2) &\approx 0.01. \end{aligned} \tag{57}$$

The argument of the function, $\alpha = \sqrt{\frac{1}{6}}x - \frac{5}{6}t$, implies that $\rho(x,t)$ behaves like a wave front moving from $-\infty$ to $+\infty$ with speed

$$u = \frac{\frac{5}{6}}{\sqrt{\frac{1}{6}}} = \frac{5}{\sqrt{6}}. \tag{58}$$

Behind the front the density is approximately equal to 1, and in front of the front approximately like 0.

If we look to at the function in $x = 0$, we can compute how long time it takes for a value $\rho = 0.01$ to reach 0.99:

$$\begin{aligned} -\frac{5}{6}t_1 &= 2.2, \\ -\frac{5}{6}t_2 &= -5.3, \end{aligned} \tag{59}$$

That is,

$$\Delta t = t_2 - t_1 = 6\frac{6}{5} - 2.2\frac{6}{5} \approx 4.5. \tag{60}$$

With a given value for $r = 0.3\text{year}^{-1}$ and time scale r^{-1}, it becomes

$$\Delta t \approx \frac{4.5}{r} = 15\text{years} \tag{61}$$