Norwegian University of Science and Technology
Department of Mathematical Sciences

For questions during the exam:
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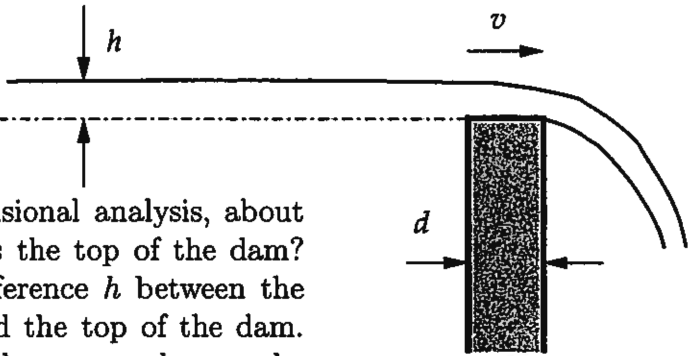
Exam in TMA4195 Mathematical modelling

Friday 17 December 2004
Duration: 4 hours

Permitted aids (code C): Simple calculator (HP 30S)
Rottman: *Matematisk formelsamling*

Grades available: 17 January 2005



Problem 1 When water behind a dam rises higher than the top of the dam, the water will start running over the top of the dam. What can you tell, just using dimensional analysis, about the mean velocity of the water flow across the top of the dam? Assume that it depends on the height difference h between the water (some distance behind the dam) and the top of the dam. Assume also that the thickness d of the dam may play a role, together with the density ρ and the kinematic viscosity ν (measured in m^2/s) of the water. It is reasonable to assume that the answer has a finite limit when $d \to 0$ and $\nu \to 0$. What does dimensional analysis say about this limit?

Problem 2 Draw a bifurcation diagram showing the position and stability for the equilibrium points of the system

$$\frac{du}{dt} = (\mu - 2u^2 + u^3)(\mu + u).$$

Problem 3

a) What is the difference between a regular and a singular perturbation problem?

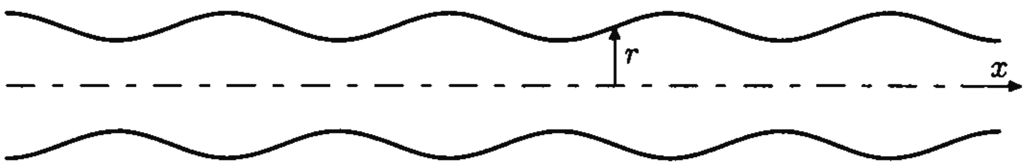
b) Consider the problem

$$y'' = (1 + \varepsilon x)y, \quad y(0) = 1, \quad y'(0) = -1.$$

Find an approximate solution with an error $O(\varepsilon^2)$ for $0 \le x \le 1$.
(You don't need to *prove* any error estimate. Hint: Try $y_1 = wy_0$ for a function w.)

Problem 4 Imagine the following experiment performed in a situation of negligible gravity – for example on a space station: We create a cylindrical "sausage" of water, as much at rest as we can manage, and see how it develops. (We ignore the not inconsiderable difficulties in managing this in practice.)

The water "sausage" cannot have a completely uniform radius. Where the radius is smaller, the pressure will be greater because of the surface tension, and the resulting pressure gradient will drive the water towards areas of lower pressure, i.e., where the "sausage" is already thicker. This instability will eventually lead to the "sausage" breaking into a number of isolated drops.



(One sees the same phenomenon in water slowly leaking from a water spout, but gravity makes that problem more difficult to analyse.)

a) Assume that the time T it takes for the water to break into drops depends on the average radius R of the water "sausage", the density ρ of water, and the surface tension σ (with units $\mathrm{N/m} = \mathrm{kg/s^2}$). What is the most general form of this dependence?

The velocity field $\mathbf{v}^*$ in the water will satisfy the *Navier–Stokes equation*:

$$\rho\frac{\partial \mathbf{v}^*}{\partial t^*} + \rho(\mathbf{v}^* \cdot \nabla_*)\mathbf{v}^* = -\nabla_* p^* + \mu\nabla_*^2\mathbf{v}^*, \tag{1}$$

where ∇_* is the gradient with respect to $\mathbf{x}^*$. We consider the density ρ a constant. Additionally, the pressure (with the atmospheric pressure subtracted) will satisfy the boundary condition

$$p^* = \sigma\kappa^*$$

on the surface, where κ^* is the *mean curvature* of the surface. (You don't need to know the exact definition of κ^*, only that it is measured in m^{-1} and scales the way you would expect an inverse length to scale when the length scale is changed. But it may be useful to know that $\kappa^* = 1/r^*$ for a sphere, and $\kappa^* = 1/(2r^*)$ for a cylinder, with radius r^*.)

b) Introduce a scaling with a length scale R (the mean radius of the water "sausage") and pressure scale $\sigma/(2R)$. Call the time scale T, and choose a consistent scaling for $\mathbf{v}^*$. What is a reasonable value of T, if you assume that the viscous term (the final term on the right-hand side) in (1) is negligible? Compare this with the answer to the previous question. Write up the scaled version of (1). Under what circumstance does the assumption that the viscous term is negligible appear to be reasonable? Express the answer both in general terms and specifically for water, with $\mu = 10^{-3}\,\mathrm{kg\,m/s}$ and $\sigma = 0.07\,\mathrm{N/m}$.

We shall assume that the water fills the region given by $\sqrt{y^2+z^2} \le r(x,t)$, and that the x component u of $\mathbf{v}$ is a function of x alone.

c) Give an argument justifying the equation

$$\frac{\partial}{\partial t}(r^2) + \frac{\partial}{\partial x}(r^2 u) = 0. \tag{2}$$

If we now move one step further and write $\mathbf{v} = (u, 0, 0)$, then (1) is reduced – after scaling, and after throwing away the viscous term – to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial}{\partial x}\Big(\frac{1}{r} - \frac{\partial^2 r}{\partial x^2}\Big). \tag{3}$$

(You are not expected to derive this equation. The expression in parentheses is approximately proportional to the mean curvature when $|\partial r/\partial x| \ll 1$ – and thus represents the dimensionless pressure.)

d) What is the linearisation of the system (2), (3) around the equilibrium solution $u = 0$, $r = 1$? (Set $u = \tilde{u}$ and $r = 1 + \tilde{r}$ and find linear equations for $\tilde{u}$, $\tilde{r}$, approximately valid when $|\tilde{u}| \ll 1$ and $|\tilde{r}| \ll 1$.)

Investigate the stability of the equilibrium solution $u = 0$, $r = 1$ by trying solutions on the form $\tilde{u} = u_0 e^{\lambda t + ikx}$, $\tilde{r} = r_0 e^{\lambda t + ikx}$, with real k. You will find real λ, one positive and one negative, so long as k is small enough. For which k do we get the largest positive λ? What are the the physical interpretations of this value of k and the corresponding λ?