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For questions during the exam:
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English

Exam in TMA4195 Mathematical modeling

Thursday 15 desember 2005
Duration: 4 hours

Permitted aids (code C): Simple calculator (HP 30S)
Rottman: *Matematisk formelsamling*

Grades available: 15 January 2006

Problem 1 A spinning planet becomes somewhat flattened at the poles because of the rotation. The flattening is usually expressed as the difference between the equatorial radius and the polar radius, divided by the former. It is expected to depend on the universal constant of gravitation G (units $\mathrm{m^3kg^{-1}s^{-2}}$), the planet's density ρ, its volume V, and the angular velocity Ω. What can you say about this dependence based on dimensional analysis?

Problem 2 The equation of motion for an electron freely moving in an electric field can be be written in relativistic mechanics as

$$\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{c^2}}} = e\mathbf{E}$$

where m is the rest mass of the electron, e is the electric charge of the electron, $\mathbf{E}$ is the electric field, c is the speed of light, and $\mathbf{v}$ is the velocity of the electron.

Below we let $\mathbf{E} = E\mathbf{i}$ be a constant field in the x direction, and we assume that the electron starts with a velocity $\mathbf{v}(0) = u\mathbf{j}$ in the y direction, where $0 < u \ll c$.

a) Describe a suitable scaling for the early parts of the motion, and show that the equation of motion together with initial data can be written in scaled variables as

$$\frac{d}{dt}\frac{\mathbf{v}}{\sqrt{1-\varepsilon|\mathbf{v}|^2}} = \mathbf{i}, \quad \mathbf{v}(0) = \mathbf{j},$$

where we as usual write $\mathbf{v}$ in place $\mathbf{v}'$, and so forth ("dropped the primes").

b) Find $\mathbf{v}_0$ and $\mathbf{v}_1$ in the perturbation expansion

$$\mathbf{v}(t) = \mathbf{v}_0(t) + \varepsilon \mathbf{v}_1(t) + O(\varepsilon^2).$$

For roughly how large t does this approximation get really bad?

Problem 3 Draw a bifurcation diagram in the (μ, x) plane for the dynamical system

$$\dot{x} = (\mu - x)(x^2 + \mu^2 - 2\mu)$$

showing the stable points of equilibrium as a solid curve, and the unstable ones as a dashed curve.

Problem 4 A method that has been used to measure climatic variations of temperature of the past thousands of years, is to drill a hole in the Greenland ice cap and measure the temperatures in the borehole. If not for heat conduction in the ice, each layer of ice would "remember" the temperature it had when it fell as snow, and one could recover a perfect temperature history in this way.

Grossly simplified we can say that the ice cap grows by a constant amount of snow falling each year, in the order of a meter. As the snow is buried in more snow, it is compacted into a roughly 30 cm thick layer of ice. We simplify further and assume that this happens instantaneously.

Let x be the distance into the ice from the top. Let the ice grow with a speed v, so that the snow that falls at time t_0 is found in $x = v \cdot (t - t_0)$ at time $t > t_0$. Write $u(x,t)$ for the temperature in the ice, and $U(t)$ for the surface temperature at time t.

a) Derive a model on the form

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad x > 0,$$
$$u(0,t) = U(t)$$

and briefly describe the significance of each term in the differential equation and what assumptions lie behind the model.

b) For a given time scale T the model can be rescaled in the form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} \qquad x > 0,$$
$$u(0,t) = U(t),$$

where we have already "dropped the primes". What is ε in this equation? What is the solution of the problem if $\varepsilon = 0$?

The coefficient of diffusion for heat in ice is $\kappa \approx 1000\,\mathrm{m}^2/\mathrm{Gs}$. (1 Gs – one gigasecond – is somewhat less than 32 years.) For the Greenland icecap we can set $v \approx 10\,\mathrm{m}/\mathrm{Gs}$. For roughly what values of the time scale T will heat conduction play an important role?

The problem comes with a "built in" time scale and a length scale, which appear by scaling the equation to balance all three terms in the differential equation. What are these time and length scales for our model of the Greenland icecap?

(Do not try to *solve* the equations in this problem for $\varepsilon > 0$, not even using perturbation methods. A thorough analysis of this problem will require more time than we have available here.)

Problem 5 *As you can see, this problem comes with a lot of text. There is not so very much computation to be done, but there are quite a number of things to keep track of. This is obviously the "hard" part of the problem set. It is probably not a good idea to begin on this problem until you have the rest of the problems under control.*

We now return to the flattened planet from problem 1. After posing and scaling the problem, we are left with the following, where we assume that the planet has uniform density and rotates around the z axis:

We use spherical coordinates ($x = r\cos\theta\sin\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\varphi$). Because of rotational symmetry about the z axis, the angle θ does not appear in any functions, so we use the notation $V(r,\varphi)$ etc. The interior of the planet is given by $r \le F(\varphi)$, where

$$\int_0^\pi F(\varphi)^3 \sin\varphi\, d\varphi = 2 \tag{1}$$

(which corresponds to the volume of the planet being $\frac{4}{3}\pi$). The potensial V in the gravitaional field satisfies

$$\nabla^2 V = \begin{cases} 3 & r < F(\varphi), \\ 0 & r > F(\varphi). \end{cases} \tag{2}$$

It will suit our purposes to divide the gravitational potential in two functions, an "inside" part V_i for $r \le F(\varphi)$ and an "outside" part V_u for $r \ge F(\varphi)$. The natural continuation of (2) for $r = F(\varphi)$ is

$$V_i\big(F(\varphi),\varphi\big) = V_u\big(F(\varphi),\varphi\big), \qquad \frac{\partial V_i}{\partial r}\big(F(\varphi),\varphi\big) = \frac{\partial V_u}{\partial r}\big(F(\varphi),\varphi\big). \tag{3}$$

Furthermore, $\lim_{r\to\infty} V = 0$ and

$$V\big(F(\varphi),\varphi\big) - \tfrac{1}{2}\varepsilon F(\varphi)^2 \sin^2\varphi = \text{constant} \tag{4}$$

(This corresponds to the surface being locally "horizontal" when the centrifugal force is accounted for. For the Earth, $\varepsilon \approx 0.00345$. In general, ε is small.)

When $\varepsilon = 0$ you may assume as a given that the problem has the solution $F(\varphi) = 1$ and $V = V_0$, where

(5) $$V_0 = \begin{cases} V_{i0} = \frac{1}{2}r^2 - \frac{3}{2} & r \le 1, \\ V_{u0} = -\dfrac{1}{r} & r \ge 1. \end{cases}$$

When $0 < \varepsilon \ll 1$ it is natural to assume that we can write

$$F(\varphi) = 1 + \varepsilon f(\varphi) + O(\varepsilon^2), \quad V = V_0 + \varepsilon V_1 + O(\varepsilon^2).$$

The difficulty here is that the coupling condition (3) is given on the planetary surface, which is unknown. The solution of this difficulty is to extrapolate V_i and V_u to $r = 1$ and write an adjusted coupling condition there instead. We use Taylor's formula:

$$V_i\bigl(1 + \varepsilon f(\varphi), \varphi\bigr) = V_i(1, \varphi) + \varepsilon f(\varphi)\frac{\partial V_i}{\partial r}(1, \varphi) + O(\varepsilon^2)$$

and correspondingly for V_u. Insert $V_i = V_{i0} + \varepsilon V_{i1} + O(\varepsilon^2)$ and correspondingly for V_u. Use the solution (5) for V_0 and show that $V_{i1}(1, \varphi) = V_{u1}(1, \varphi)$.

Use Taylor's formula for $\partial V/\partial r$ the same way, and explain how that leads to

(6) $$\frac{\partial V_{u1}}{\partial r}(1, \varphi) - \frac{\partial V_{i1}}{\partial r}(1, \varphi) = 3f(\varphi).$$

Next, show using (1) that

(7) $$\int_0^\pi f(\varphi) \sin\varphi \, d\varphi = 0,$$

and using (2) that $\nabla^2 V_{i1} = \nabla^2 V_{u1} = 0$. Show by extrapolation to $r = 1$ that (4) yields

(8) $$V_1(1, \varphi) + f(\varphi) - \tfrac{1}{2}\sin^2\varphi = \text{constant}.$$

Finally, show that there is a solution on the form

$$V_{i1} = ar^2 P_2(\cos\varphi),$$
$$V_{u1} = ar^{-3} P_2(\cos\varphi),$$

and determine a and $f(\varphi)$.

Formulas you may take for granted: The Laplace operator in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial V}{\partial\varphi}\right) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 V}{\partial\theta^2}.$$

P_2 is the Legendre polynomial $P_2(w) = \frac{3}{2}w^2 - \frac{1}{2}$. Both $r^2 P_2(\cos\varphi)$ and $r^{-3} P_2(\cos\varphi)$ satisfy the Laplace equation $\nabla^2 V = 0$.