TMA4195 Mathematical modeling, 15. december 2005

Suggested solution

Problem 1

We can put up the dimension matrix:

	G	ρ	V	Ω
kg	-1	1	0	0
m	3	-3	3	0
s	-2	0	0	-1

It has rank 3 (consider the last three columns, for example). Thus the null space has dimension $4-3=1$, so there is just one independent combination: For example, $\Omega^2/(G\rho)$. (This is easily found without writing up the dimension matrix, and it is not hard to give a direct convincing argument that this is essentially the only dimensionless combination.)

We neglegted to mention the flattening, which is of course already dimensionless. By Buckingham's pi theorem the flattening must be a function of $\Omega^2/(G\rho)$.

Problem 2

a) The electron starts with a moderate (relative to c) velocity $|\mathbf{v}| = u$ and an approximate acceleration of magnitude eE/m. The velocity scale u seems reasonable to begin with. At the given acceleration an additional velocity u is gained over a time $e = u/(eE/m) = mu/(eE)$, so we pick this as our time scale: Thus we write

$$\mathbf{v} = u\mathbf{v}', \quad t = \frac{mu}{eE}t'$$

and get

$$\frac{eE}{mu}\frac{d}{dt'}\frac{mu\mathbf{v}'}{\sqrt{1-\frac{\left(u|\mathbf{v}'|^2\right)}{c^2}}} = eE\mathbf{i}$$

which, after canceling various terms and dropping the primes, becomes

$$\frac{d}{dt}\frac{\mathbf{v}}{\sqrt{1-\varepsilon|\mathbf{v}|^2}} = \mathbf{i}, \qquad \varepsilon = \frac{u^2}{c^2}.$$

The initial condition $\mathbf{v}(0) = u\mathbf{j}$ is $u\mathbf{v}'(0) = u\mathbf{j}$, which becomes $\mathbf{v}(0) = \mathbf{i}$ after division by u and dropping the prime.

b) By an application of Taylor's formula (or the binomial theorem) we write

$$\frac{1}{\sqrt{1-\varepsilon|\mathbf{v}|^2}} = 1 + \tfrac{1}{2}\varepsilon|\mathbf{v}|^2 + O(\varepsilon^2)$$

so that we can write

$$\frac{d}{dt}\big((1+\tfrac{1}{2}\varepsilon|\mathbf{v}|^2)\mathbf{v}\big) = \mathbf{i} + O(\varepsilon^2).$$

Now, with $\mathbf{v} = \mathbf{v}_0 + \varepsilon\mathbf{v}_1 + O(\varepsilon^2)$ we find $|\mathbf{v}|^2 = |\mathbf{v}_0|^2 + O(\varepsilon)$. Thus we get

$$\frac{d}{dt}\big((1+\tfrac{1}{2}\varepsilon|\mathbf{v}_0|^2)\mathbf{v}_0 + \varepsilon\mathbf{v}_1\big) = \mathbf{i} + O(\varepsilon^2),$$

or more usefully

$$\frac{d\mathbf{v}_0}{dt} + \varepsilon\frac{d}{dt}\big(\mathbf{v}_1 + \tfrac{1}{2}|\mathbf{v}_0|^2\mathbf{v}_0\big) = \mathbf{i} + O(\varepsilon^2),$$

which leads to

$$\frac{d\mathbf{v}_0}{dt} = \mathbf{i}, \quad \frac{d}{dt}\big(\mathbf{v}_1 + \tfrac{1}{2}|\mathbf{v}_0|^2\mathbf{v}_0\big) = 0.$$

The first of these, with the initial condition $\mathbf{v}_0(0) = \mathbf{j}$, is immediately integrated to get

$$\mathbf{v}_0 = t\mathbf{i} + \mathbf{j}.$$

The second, with the initial condition $\mathbf{v}_1(0) = 0$, yields

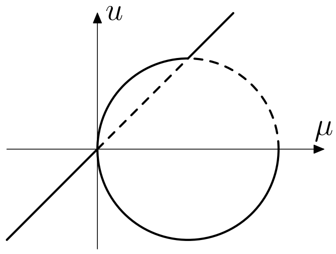
$$\mathbf{v}_1 = \tfrac{1}{2}\big(\mathbf{j} - |\mathbf{v}_0|^2\mathbf{v}_0\big) = \tfrac{1}{2}\big(\mathbf{j} - (t^2+1)(t\mathbf{i} + j)\big) = \tfrac{1}{2}\big(-(t^3+t)\mathbf{i} + t^2\mathbf{j}\big).$$

The total approximation is

$$\mathbf{v} = t\mathbf{i} + \mathbf{j} + \tfrac{1}{2}\varepsilon\big(-(t^3+t)\mathbf{i} + t^2\mathbf{j}\big).$$

The "correction term" gains the same magnitude as the lowest order term at time $t \sim 1/\sqrt{\varepsilon}$. At this time the total velocity v is about $1/\sqrt{\varepsilon}$, which in unscaled variables becomes $u/\sqrt{\varepsilon} = c$. We should not be surprised that our approximation, based on velocities $\ll c$, becomes inaccurate when the velocities approach c.

Problem 3



Problem 4

a) Note that the ice is moving with a velocity v in the (x,t) coordinate system. For a moving control volume $a < x < b$ the conservation of energy becomes

$$\frac{d}{dt}\int_{a(t)}^{b(t)} \rho c u\,dx = k\frac{\partial u}{\partial x}(b,t) - k\frac{\partial u}{\partial x}(a,t).$$

where ρ is the density, c the specific heat capacity, and k the heat conductivity of the ice. This is assuming that no other processes in the ice disturb the energy balance. Use the transport theorem on the integral (with $\dot{a} = \dot{b} = v$), and also assume sufficient smoothness of u so the differentiation can be taken inside the integral sign, and get

$$\int_a^b \rho c\frac{\partial u}{\partial t}\,dx + v\rho c u(b,t) - v\rho c u(a,t) = k\frac{\partial u}{\partial x}(b,t) - k\frac{\partial u}{\partial x}(a,t).$$

Next rewrite the differences as integrals of derivatives:

$$\int_a^b \rho c \frac{\partial u}{\partial t}\,dx + \int_a^b v\rho c \frac{\partial u}{\partial x}\,dx = \int_a^b k \frac{\partial^2 u}{\partial x^2}\,dx$$

Finally, collect all the integrals into one integral, use the arbitrariness of a and b to show that the integrand must be zero, and divide by ρc to arrive at the given equation, where $\kappa = k/(\rho c)$.

The condition $u(0,t) = U(t)$ is just a restatement of the fact that the surface temperature is U, and a recognition of the fact that the surface is always at $x = 0$.

Alternatively, one could get the usual heat conduction equation in a coordinate system where the ice does not move and then change coordinates.

b) Setting $t = Tt'$ and $x = vTx'$ in the model, then "dropping the primes" results in the rescaled model in the problem, with

$$\varepsilon = \frac{\kappa}{v^2 T}.$$

For $\varepsilon = 0$ the problem becomes

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad x > 0,$$
$$u(0,t) = U(t),$$

which is a simple transport equation with solution

$$u(x,t) = U(x-t).$$

(If one wishes to see it in the usual conservation law terms, the characteristic equation is $\dot{x} = 1$, with solution $x = t + t_0$. So the characteristic through a given point (x,t) starts at $(0,t_0)$ with $t_0 = x - t$. Hence the formula above. The setup is a bit unusual in that conditions are given on the boundary $x = 0$, rather than at an initial moment $t = 0$.)

Write $\tau = \kappa/v^2$. From the given values we find $\tau \approx 10\,\mathrm{Gs}$, or not quite 320 years. Since $\varepsilon = \tau/T$, we conclude that heat conduction is important when $T \approx \tau$ or less; that is, on time scales of roughly 320 years or less.

Our scaling already balances the first two terms of the equation. To balance the third term with the other two, all we need to do is put $\varepsilon = 1$. This implies $T = \tau$. The corresponding length scale is $v\tau \approx 100\,\mathrm{m}$.

Problem 5

Setting $V_i = V_{i0} + \varepsilon V_{i1} + O(\varepsilon^2)$ in Taylor's formula

$$V_i\big(1 + \varepsilon f(\varphi), \varphi\big) = V_i(1,\varphi) + \varepsilon f(\varphi)\frac{\partial V_i}{\partial r}(1,\varphi) + O(\varepsilon^2)$$

and collecting equal powers of ε we find

$$V_i\big(1 + \varepsilon f(\varphi), \varphi\big) = V_{i0}(1,\varphi) + \varepsilon\Big(V_{i1}(1,\varphi) + f(\varphi)\frac{\partial V_{i0}}{\partial r}(1,\varphi)\Big) + O(\varepsilon^2).$$

Now doing the same for V_u we compare terms, and get

$$V_{i0}(1,\varphi) = V_{u0}(1,\varphi), \quad V_{i1}(1,\varphi) + f(\varphi)\frac{\partial V_{i0}}{\partial r}(1,\varphi) = V_{u1}(1,\varphi) + f(\varphi)\frac{\partial V_{u0}}{\partial r}(1,\varphi).$$

The equation on the left is already satisfied. But also, the two partial derivatives in the equation on the right are equal (from looking at the formulas for V_0), so that $V_{i1}(1,\varphi) = V_{u1}(1,\varphi)$ follows.

Next, Taylor's formula for $\partial V_i/\partial r$ is

$$\frac{\partial V_i}{\partial r}\bigl(1+\varepsilon f(\varphi),\varphi\bigr) = \frac{\partial V_i}{\partial r}(1,\varphi) + \varepsilon f(\varphi)\frac{\partial^2 V_i}{\partial r^2}(1,\varphi) + O(\varepsilon^2).$$

Once more we insert $V_i = V_{i0} + \varepsilon V_{i1} + O(\varepsilon^2)$ and collect equal powers:

$$\frac{\partial V_i}{\partial r}\bigl(1+\varepsilon f(\varphi),\varphi\bigr) = \frac{\partial V_{i0}}{\partial r}(1,\varphi). + \varepsilon\Bigl(\frac{\partial V_{i1}}{\partial r}(1,\varphi) + f(\varphi)\frac{\partial^2 V_{i0}}{\partial r^2}(1,\varphi)\Bigr) + O(\varepsilon^2)$$

Doing the same for V_u and comparing terms we get

$$\frac{\partial V_{i0}}{\partial r}(1,\varphi) = \frac{\partial V_{u0}}{\partial r}(1,\varphi), \quad \frac{\partial V_{i1}}{\partial r}(1,\varphi) + f(\varphi)\frac{\partial^2 V_{i0}}{\partial r^2}(1,\varphi) = \frac{\partial V_{u1}}{\partial r}(1,\varphi) + f(\varphi)\frac{\partial^2 V_{u0}}{\partial r^2}(1,\varphi).$$

Again, the equation on the left is already satisfied. The equation on the right is

$$\frac{\partial V_{i1}}{\partial r}(1,\varphi) + f(\varphi) = \frac{\partial V_{u1}}{\partial r}(1,\varphi) - 2f(\varphi),$$

which gives (6).

Equation (1) in the problem gives

$$\begin{aligned} 2 &= \int_0^\pi \bigl(1+\varepsilon f(\varphi) + O(\varepsilon^2)\bigr)^3 \sin\varphi\, d\varphi = \int_0^\pi \bigl(1+3\varepsilon f(\varphi)\bigr)\sin\varphi\, d\varphi + O(\varepsilon^2) \\ &= 2 + 3\varepsilon\int_0^\pi f(\varphi)\sin\varphi\, d\varphi + O(\varepsilon^2) \end{aligned}$$

which implies (7). Similarly

$$3 = \nabla^2 V_i = \nabla^2 V_{i0} + \varepsilon\nabla^2 V_{i1} + O(\varepsilon^2), \quad 0 = \nabla^2 V_u = \nabla^2 V_{u0} + \varepsilon\nabla^2 V_{u1} + O(\varepsilon^2)$$

imply $\nabla^2 V_{i1} = \nabla^2 V_{u1} = 0$. Equation (4) becomes

$$V(1,\varphi) + \varepsilon f(\varphi)\frac{\partial V}{\partial r}(1,\varphi) - \tfrac{1}{2}\varepsilon\bigl(1+\varepsilon f(\varphi)\bigr)^2 \sin^2\varphi = \text{constant}.$$

Inserting $V = V_0 + \varepsilon V_1$ and using $V_0(1,\varphi) = 1$ and $\partial V_0/\partial r(1,\varphi) = 1$ we are left with

$$1 + \varepsilon V_1(1,\varphi) + \varepsilon f(\varphi) - \tfrac{1}{2}\varepsilon\sin^2\varphi + O(\varepsilon^2) = \text{constant},$$

which is clearly satisfied to lowest order, while the $O(\varepsilon)$ terms add up to (8.) It clearly does not matter here whether we use V_{i1} or V_{u1}.

Now we try to satisfy (6):

$$3f(\varphi) = \frac{\partial V_{u1}}{\partial r}(1,\varphi) - \frac{\partial V_{i1}}{\partial r}(1,\varphi) = -5aP_2(\cos\varphi).$$

Thus

$$V_1(1,\varphi) + f(\varphi) - \tfrac{1}{2}\sin^2\varphi = -\tfrac{2}{3}aP_2(\cos\varphi) - \tfrac{1}{2}\sin^2\varphi = -a\cos^2\varphi + \tfrac{1}{3}a - \tfrac{1}{2}\sin^2\varphi$$

which is a constant precisely when $a = \frac{1}{2}$. Thus we end up with

$$f(\varphi) = -\tfrac{5}{6}P_2(\cos\varphi).$$

The flatting is $-\frac{5}{6}\big(P_2(0)+P_2(1)\big)\varepsilon+O(\varepsilon^2)=\frac{5}{4}\varepsilon+O(\varepsilon^2)$. For the Earth, the leading term is ≈ 0.0043. The correct flattening of the Earth is 0.0036. The discrepancy is probably because the earth does not have uniform density, but has a heavy iron core instead.

Addendum. Here I present the modeling leading up to problem 5. This is not a part of the exam solution, but is provided for completeness' sake.

Let the planet have a uniform density ρ and a volume V, so that its mass is $M=\rho V$ and its radius, if it is not spinning and hence forms a perfect sphere, is R where $\frac{4}{3}\pi R^3=V$. When the planet *is* rotating at moderate speeds, it will be near a sphere of this radius, and we write the surface of the spinning planet as $r=RF(\varphi)$ in spherical coordinates.

Writing up the volume of the planet as an integral in spherical coordinates, and requiring that this volume is still V, we arrive at the condition

$$\int_0^\pi F(\varphi)^3\sin\varphi\,d\varphi=2.$$

The fundamental equation of the gravitational potential is $\nabla^2 V=4\pi G\rho$, which in this case becomes

$$\nabla^2 V=\begin{cases}4\pi G\rho & r<RF(\varphi),\\ 0 & r>RF(\varphi),\end{cases}$$

with the additional requirement that V and its first order derivatives are continuous across the planetary surface $r=RF(\varphi)$. (Otherwise, a sort of mass distribution like a Dirac delta is indicated on the surface.)

A planet typically behaves like a blob of liquid: Its surface will be an equipotential surface. It it is spinning with angular velocity Ω around the z axis then we will observe a centrifugal force in a coordinate system that is fixed with respect to the planet. This force is in fact conservative, with potential $-\frac{1}{2}\Omega^2(x^2+y^2)$. Therefore,

$$V-\tfrac{1}{2}\Omega^2(x^2+y^2)=\text{constant}\qquad\text{on the surface } r=RF(\varphi).$$

The solution for $\Omega=0$ is $F(\varphi)=1$ and

$$V=\begin{cases}-\dfrac{MG}{r} & r\ge R,\\ \frac{2}{3}\pi G\rho r^2+\text{constant} & r\le R\end{cases}$$

where the constant is chosen to make V continuous at $r=R$.

The obvious scaling for the problem uses the length scale R. Equally obvious is the scale MG/R for V. These scalings immediately leads to the problem on scaled form as given in the exam problem, with

$$\varepsilon=\frac{\Omega^2R^3}{MG}=\frac{3}{4}\frac{\Omega^2}{\pi G\rho},$$

(notice the connection with problem 1), or alternatively

$$\varepsilon=\frac{\Omega^2R}{MG/R^2}=\frac{\Omega^2R}{g}$$

where the numerator Ω^2R is the centrifugal acceleration at the equator, while the denominator g is the acceleration of gravity on the surface. Clearly, it will be a very unusual planet that does not have $\varepsilon\ll 1$.

Above, I blamed the difference between my solution and the actual flattening of the Earth on the heavy core. If we consider the extreme case of a core that is just a point mass and contains most of the planetary mass, our problem is reduced to the much simpler of computing the equipotential surface near $r=1$ of the potential

$$-\frac{1}{r}-\tfrac{1}{2}\varepsilon(x^2+y^2).$$

On the z axis, this potential is just $-1/r$, which equals -1 at $r=1$. In the xy plane, on the other hand, the potential is $-1/r-\frac{1}{2}\varepsilon r^2$. Setting $r=1+\delta$, this is approximately $-1+\delta-\frac{1}{2}\varepsilon$, which is -1 for $\delta=\frac{1}{2}\varepsilon$. Thus the flattening in this case is only about $\frac{1}{2}\varepsilon$. Certainly, therefore, it is reasonable to expect less flattening than predicted by our model when the planetary core is heavier, a condition that surely is shared by all planets.