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TMA4195 Mathematical Modelling

Date: Wednesday, December 12, 2007
Time: 09.00 - 13:00
Aids: (Code C): Simple calculator (HP 30S) Rottman: Matematisk formelsamling
(Norwegian or English)
English

Grading finished: January 12, 2008

Problem 1 Air resistance, F, of a car depends on its length, L, cross sectional area, A, its speed relative to the air, U, the density of the air, ρ, and the air's kinematic viscosity, ν.

a) Use dimensional analysis to derive the equation

$$F = \rho U^2 A \phi \left(\frac{UL}{\nu}, \frac{A}{L^2} \right). \tag{1}$$

In order to determine the function ϕ, the engineers have suggested to test 1:10 scale models in the long water tank at the Tyholt model basin by dragging them through water (5×10m cross section, 270m length).

b) Is this a good idea?

(For estimates: Air: $\nu = 10^{-5}\text{m}^2/\text{s}$ and $\rho = 1\text{kg/m}^3$. Water: $\nu = 10^{-6}\text{m}^2/\text{s}$ and $\rho = 10^3\text{kg/m}^3$).

Problem 2 Determine the equilibrium points and whether they are stable or unstable for the following equation:

$$\frac{du}{dt} = \left(u - u^2\right)\left(u - \mu\right), \quad u \geq 0, \mu \geq 0. \tag{2}$$

Problem 3 The cell density, n^*, in a part of the body may be modelled as

$$\frac{dn^*}{dt^*} = \alpha n^* - \omega n^*, \tag{3}$$

where α is the birth rate and ω the death rate. In order to prevent that the density runs astray, the cells produce a so-called *inhibitor* which dampens uncontrolled growth. The inhibitor has density c^* and works by changing the the birth rate to

$$\alpha = \frac{\alpha_0}{1 + c^*/A}. \tag{4}$$

The production of the inhibitor is proportional with n^*, while it breaks down by the rate δ:

$$\frac{dc^*}{dt^*} = \beta n^* - \delta c^*. \tag{5}$$

This system has a time scale ω^{-1} connected to the breakdown of the cells, and a time scale δ^{-1} connected to the breakdown of the inhibitor. It is known that $\omega^{-1} \gg \delta^{-1}$.

a) Scale the system by applying ω^{-1} as the time scale and A as a scale for c^*. Show that the system with a certain scale for n^* may be written

$$\begin{aligned} \dot{n} &= \left(\frac{\kappa}{1+c} - 1\right) n, \\ \varepsilon \dot{c} &= n - c. \end{aligned} \tag{6}$$

What is the meaning of ε and κ? What may be said about the size of ε, and what is such a system called? Determine what kind of equilibrium point the trivial equilibrium point $(0,0)$ is. (Here and below we assume that κ is somewhat larger than 1).

b) Determine the path and the equation for the motion of the outer solution of Eqn. (6) to leading order ($O(1)$). Show, without necessarily solving the differential equation that all motion on this path converges to an equilibrium point for the full system.

c) Determine to leading order the inner solution of (6) by introducing a new time scale. Then determine a uniform, approximate solution (It is not possible to solve the equation in (b) explicitly).

Problem 4

a) Define the scaled flux and kinematic velocity in the standard model for road traffic, which leads to the differential equation:

$$\rho_t + (1 - 2\rho)\,\rho_x = 0. \tag{7}$$

Sketch the characteristics and the solution $\rho(x,t)$ to Eqn. (7) for $t > 0$ if

$$\begin{aligned} (i) \quad \rho(x,0) &= \begin{cases} 1 & x < 0, \\ 0 & x \geq 0. \end{cases} \\ (ii) \quad \rho(x,0) &= \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases} \end{aligned} \tag{8}$$

We are from now considering a situation where cars are continuously entering and leaving the road (the road itself is a one-way street). This will be modelled as a source term, such that the equation becomes

$$\rho_t + (1 - 2\rho)\,\rho_x = \varepsilon\left(\frac{1}{2} - \rho\right), \quad \varepsilon > 0. \tag{9}$$

(If $\rho < \frac{1}{2}$, there is a net influx of cars, whereas cars are leaving the road if $\rho > \frac{1}{2}$).

b) Show that a characteristic curve starting at $(0, x_0, c_0)$ may, for $t \geq 0$, be written as

$$\left\{t\ ,\ x_0 + \frac{(1-2\rho_0)}{\varepsilon}\left(1 - e^{-\varepsilon t}\right)\ ,\ \frac{1}{2} + \left(\rho_0 - \frac{1}{2}\right)e^{-\varepsilon t}\right\}. \tag{10}$$

c) Find the solution of Eqn. (9) for $t > 0$ with (i) in (8) as initial condition.

d) Show that the solution of Eqn. (9) for $t > 0$ with (ii) in (8) as initial condition develops a shock. Use the conservation law to argue that the location of the shock may be stationary. Assuming this, determine the solution.

(Hint: The equation

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} - R(x,y,z) = 0$$

has the following equations for the characteristics

$$\frac{dx}{ds} = P(x,y,z)\,,\ \frac{dy}{ds} = Q(x,y,z)\,,\ \frac{dz}{ds} = R(x,y,z)\ ).$$