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EXAM IN TMA4195 Mathematical Modeling

English
Tuesday, December 21, 2010
9:00 – 13:00

Aids (code C): Approved calculator
Rottman: *Matematisk formelsamling*

Results: January 21, 2011

Problem 1 Let $0 < \alpha < r$, $N_0 > 0$, and consider the following initial value problem,

$$(1)\qquad \begin{cases} \dfrac{dN^*(t^*)}{dt^*} = rN^*(t^*)\Big(1 - \dfrac{N^*(t^*)}{K}\Big) - \alpha N^*(t^*), \quad t^* > 0, \\ N^*(0) = N_0. \end{cases}$$

a) What could problem (1) model?

Find reasonable scales for this problem when $N_0 \gg K$ and $\alpha < r$.

Let $\epsilon, \alpha, \beta > 0$, $\alpha, \beta < 1$, and consider the following scaled system of equations,

$$(2)\qquad \begin{cases} \dfrac{dx}{dt} = x(1-x) - \alpha xy, \\ \epsilon\dfrac{dy}{dt} = y(1-y) - \beta xy. \end{cases}$$

b) What could system (2) model?

Find the equilibrium points and determine their stability when $\beta = 0$.

Problem 2 In statistical thermodynamics the equation of state of a closed non-relativistic system of N particles of mass m has the form

$$H = \Phi(p, S, m, N, h, k).$$

Here H is the enthalpy, p pressure, V volume, S entropy, and k and h are the Boltzmann and Planck constants. The dimensions are $[p] = \mathrm{kg\,m^{-1}\,s^{-2}}$, $[H] = \mathrm{kg\,m^2\,s^{-2}}$, $[S] = [k] = \mathrm{kg\,m^2\,s^{-2}\,K^{-1}}$, and $[h] = \mathrm{kg\,m^2\,s^{-1}}$.

What can you say about this relation using dimensional analysis?

Problem 3 Consider the following differential equation

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + e^y = 0, \quad 0 < x < 1, \quad 0 < \epsilon \ll 1,$$

with boundary conditions

$$y(0) = 1, \qquad y(1) = -\ln 2.$$

Find a leading order singular perturbation approximation of y valid for all $x \in [0, 1]$.

Hint: The boundary layer is at $x = 0$.

Problem 4 In spring 2010, the Icelandic volcano Eyjafjallajökull had several eruptions producing large ash clouds. These clouds disrupted air traffic across Europe because air space had to be shut down where ash consentrations became too high.

Typically the ash clounds are relatively thin, so we will consider a 2 dimensional model of the spread of ash neglecting the vertical direction. The ash consentration is denoted by $c^* = c^*(x^*, y^*, t^*)$ ($[c]$ = number of particles/volume). We assume that the volcano is situated at $(x^*, y^*) = (0, 0)$ and that all the ash, N_0 particles, was produced in a big eruption at $t^* = 0$. At later times ($t^* > 0$) there is no production of ash. Ash will spread by difusion and convection (wind), and eventually fall to the ground due to gravity. For simplicity, we assume that the number of ash particles falling to the ground per time and area is proportional to the consentration with a proportionality factor r ($[r]$ = number of particles/(time · area · consentration)). Let D denote the diffusion coefficient and $\vec{u} = (u_1, u_2)$ the wind velocity.

a) What does Fick's law for diffusion say?

Write down the diffusive, convective, and total fluxes in this problem.

For $t^* > 0$, set up the conservation law for c^* in integral form in a region $\Omega \subset \mathbb{R}^2$.

b) Assume that c^* is smooth and show that it has to satisfy

$$\frac{\partial c^*}{\partial t^*} - \frac{\partial}{\partial x^*}\Big(D\frac{\partial c^*}{\partial x^*}\Big) - \frac{\partial}{\partial y^*}\Big(D\frac{\partial c^*}{\partial y^*}\Big) + \frac{\partial}{\partial x^*}(u_1 c^*) + \frac{\partial}{\partial y^*}(u_2 c^*) + rc^* = 0 \qquad \text{for} \quad t^* > 0.$$

What condition must c^* satisfy at $t^* = 0$?

Problem 5 In a scaled fluid dynamics model of car traffic along a one-way one-lane road, the density of cars ρ satisfy the equation

$$\rho_t + (1 - 2\rho)\rho_x = 0$$

in any x-interval where no cars can enter or leave the road.

a) Write down the car speed $v(\rho)$ and flux $j(\rho)$ in this model.

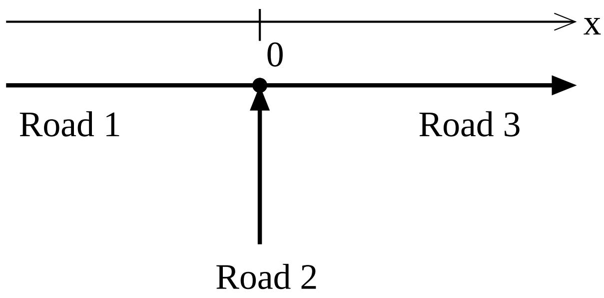
Find ρ for $x < 0$ when the following initial and boundary conditions hold:

$$\rho(x, 0) = \frac{1}{2} \qquad \text{for} \quad x < 0,$$

and

$$\rho(0, t) = \frac{1}{2} + \frac{\sqrt{2}}{4} \qquad \text{for} \quad t > 0.$$

Consider a situation where 3 one-way one-lane roads meet in a junction at $x = 0$. Traffic from Road 1 and Road 2 merge and continue onto Road 3. We assume that the car speed is



$v(\rho) = 1 - \rho$ on Road 1 and 3, that Road 2 is a priority road with a fixed constant flux j_2, and that $\rho \le \frac{1}{2}$ on Road 3 for all $t \ge 0$. The traffic is unaffected by the junction as long as the combined traffic on Road 1 and 2 does not exceed the capacity of Road 3. When this capacity is exceeded, we assume that the traffic on Road 2 remains unaffected while the traffic on Road 1 adapts in such a way that flux through the junction is maximal. It can be shown that for $x = 0$ and $t > 0$,

(3) $$j_1(0, t) + j_2 = j_3(0, t),$$

where j_1, j_2, and j_3 are the fluxes on Road 1, 2, and 3.

We now consider a situation with light traffic on Road 2 and 3, and heavy traffic on Road 1 reaching the junction at $t = 0$:

$$\rho(x,0) = \begin{cases} \frac{1}{2} & \text{for} \quad x < 0 \\ \frac{1}{8} & \text{for} \quad x \geq 0 \end{cases} \qquad \text{and} \qquad j_2 = \frac{1}{8} \quad \text{for} \quad t \geq 0.$$

b) Write down $j_1(\rho)$ and $j_3(\rho)$ and explain why (3) has to hold.
Show that $j_1(0,t) = \frac{1}{8}$ for all $t > 0$.
Hint: What is the maximal value of j_3?

c) An observer watches the road. What car speed does she observe at $x = -1$ during the time interval $t \in [0, 10]$?