Solutions to Exam in TMA4195 Mathematical Modeling December 2010

Problem 1

a) When $\alpha = 0$ equation (1) is the logistic equation for population growth with growth rate r and carrying capacity K. The α-term is a harvest or death term.

<u>Scales</u>:

Since $N_0 > K$, we see that $\dfrac{dN^*}{dt^*} < 0$ for $t \ll 1$, and hence $\max N^* = N_0$.

Take $N^* = N_0 N$, $t^* = Tt$, scale (1), and balance terms:

$$\frac{dN^*}{dt^*} = \frac{N_0}{T}\frac{dN}{dt} \overset{(1)}{=} rN_0 N\left(1 - \frac{N_0}{K}N\right) - \alpha N_0 N$$
$$\Rightarrow T = \frac{1}{r\left(1 - \dfrac{N_0}{K}\right) - \alpha} \sim \frac{K}{rN_0}$$

since $\alpha < r$ and $\frac{N_0}{K} \gg 1$.

Scales: $N^* = N_0 N$, $t^* = \frac{K}{rN_0}t$.

b) System (2) models e.g. growth of 2 populations sharing limited resource. The death-rates (α, β-terms) depend on the size of the competing population. When $\alpha, \beta = 0$, the populations experience logistic growth.

Equilibrium points, when $\beta = 0$

$$0 = \frac{dx}{dt} = f_1(x,y) = x(1-x) - \alpha xy \quad \Rightarrow x = 0 \text{ or } x = 1 - \alpha y$$
$$0 = \epsilon\frac{dx}{dt} = f_2(x,y) = y(1-y) \quad \Rightarrow y = 0 \text{ or } y = 1$$

Solutions = equilibrium points:

$$(x,y) = \{(0,0), (0,1), (1,0), (1-\alpha, 1)\}.$$

Stability:

$$\text{Jacobian} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-2x-\alpha y & -\alpha \\ 0 & 1-2y \end{bmatrix},$$

and the eigenvalues are $\lambda_1 = 1-2x-\alpha y, \lambda_2 = 1-2y$. Hence

	$(0,0)$	$(0,1)$	$(1,0)$	$(1-\alpha,1)$
λ_1	1	$1-\alpha(>0)$	-1	$-(1-\alpha)(<0)$
λ_2	1	-1	1	-1
	unstable	unstable	unstable	stable

Problem 2 The dimension matrix A is

	H	p	S	m	N	h	k
m	2	-1	2	0	0	2	2
s	-2	-2	-2	0	0	-1	-2
kg	1	1	1	1	0	1	1
K	0	0	-1	0	0	0	-1

Obs:

- rank$A = 4$, Buckingham $\Rightarrow 7-4=3$ dim.less combinations.
- $N, \frac{S}{k}$ are dim.less, and the remaining variables independent of K.

There is one remaining dim.less combination, and we try with $\frac{H}{p^a m^b h^c}$:

$$\frac{[H]}{[p]^a[m]^b[h]^c} = 1 \Rightarrow \begin{cases} 2 = -a+2c \\ -2 = -2a-c \\ 1 = a+b+c \end{cases}$$

Solution:

$$a = 2/5, \quad b = -3/5, \quad c = 6/5$$

Buckingham's π-theorem: The most general form of

$$H = \Phi(p, S, m, N, h, k)$$

is

$$\frac{H}{p^{2/5}m^{-3/5}h^{6/5}} = \Psi(N, S/k) \qquad \text{or} \qquad H = \left(\frac{p^2h^6}{m^3}\right)^{\frac{1}{5}} \Psi(N, S/k)$$

for an arbitrary function Ψ.

Problem 3

1. Outer problem ($\epsilon = 0$):

$$\frac{dy_0}{dx} + e^{y_0} = 0, \quad y_0(1) = -\ln 2$$

Separable equation:

$$\begin{aligned} &e^{-y_0} dy_0 = -dx, \quad y_0(1) = -\ln 2 \\ &\Rightarrow -e^{-y_0} = -x - 1 \\ &\Rightarrow -y_0 = \ln(x+1) \end{aligned}$$

2. Boundary layer thickness δ:
Rescale: $x = \delta s$, $y = Y$,

$$\frac{\epsilon}{\delta^2}\frac{d^2Y}{ds^2} + \frac{1}{\delta}\frac{dY}{ds} + e^Y = 0,$$

and determine the dominant scale:

i) $\epsilon \frac{d^2y}{dx^2} \sim \frac{dy}{dx} \Rightarrow \frac{\epsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \delta = \epsilon$
Check: $e^y = e^Y \sim 1 \ll \frac{1}{\epsilon} \sim \frac{\epsilon}{\delta^2}\frac{d^2Y}{ds^2} = \epsilon \frac{d^2y}{ds^2}$ ok!

ii) $\epsilon \frac{d^2y}{dx^2} \sim e^y \Rightarrow \frac{\epsilon}{\delta^2} \sim 1 \Rightarrow \delta = \sqrt{\epsilon}$
Check: $\frac{dy}{dx} = \frac{1}{\delta}\frac{dY}{ds} \sim \frac{1}{\sqrt{\delta}} \gg 1 \sim e^y$ not consistent!

3. Inner problem:
Rescale equation: $x = \delta x = \epsilon s, y = Y$

$$\begin{aligned} \frac{1}{\epsilon}\frac{d^2Y}{ds^2} + \frac{1}{\epsilon}\frac{dY}{ds} + e^Y = 0, \quad Y(0) = 1 \\ \Rightarrow Y'' + Y' + \epsilon e^Y = 0, \quad Y(0) = 1 \end{aligned}$$

Solve for $\epsilon = 0$:

$$\begin{aligned} &Y_0'' + Y_0' = 0, \quad Y_0(0) = 1 \\ &\Rightarrow Y_0 = A(e^{-s} - 1) + 1 \end{aligned}$$

4. Matching in the intermediate region:

$$\begin{aligned} &\lim_{x \to 0} y_0(x) = \lim_{s \to \infty} Y_0(s) \\ &\Rightarrow 0 = -A + 1 \Rightarrow A = 1 \\ &\Rightarrow Y_0 = e^{-s} \end{aligned}$$

5. Uniform solution:

$$\begin{aligned} y_u(x) &= y_0(s) + Y_0(x/\epsilon) - \lim_{x\to 0} y_0(x) \\ &= -\ln(x+1) + e^{-x/\epsilon} \end{aligned}$$

Problem 4

a) Fick's law: Diffusive flux $\vec{j}^*_D = -D\nabla c^*$
Fluxes:

$$\begin{aligned} \text{Diffusive}: &\quad \vec{j}^*_D = -D\nabla c^*, \\ \text{Convective}: &\quad \vec{j}^*_C = \vec{u}c^*, \\ \text{Total}: &\quad \vec{j}^* = -D\nabla c^* + \vec{u}c^*. \end{aligned}$$

General conservation law over $\Omega \subset \mathbb{R}^2$:

(1) $$\frac{d}{dt^*}\iint_\Omega c^* dx\, dy = -\int_{\partial\Omega} \vec{j}^* \cdot \vec{n}\, d\sigma + \iint_\Omega q^* dxdy$$
$$\text{change in } \Omega = \text{inflow} - \text{outflow} + \text{production}$$

where $\vec{j}^*$ is as above and

$$q^* = -rc^*$$

since there is no production at $t^* > 0$ and the loss of particles per time is proportional to c^*.

b) Obs:

i) $\frac{d}{dt^*}\iint_\Omega c^* dx\, dy = \iint_\Omega c^*_{t^*}\; dx\; dy$ if c^* is smooth enough.

ii) $\int_{\partial\Omega} \vec{j}^* \cdot \vec{n}\, d\sigma = \iint_\Omega \nabla\cdot \vec{j}^*\; dx\; dy$ by the divergence theorem.

Hence

$$(1) \Leftrightarrow \iint_\Omega c^*_{t^*} + \nabla\cdot\vec{j}^* + rc^*\, dxdy = 0,$$

and since this is true for any nice Ω in $\mathbb{R}^2$,

$$c^*_{t^*} + \nabla\cdot\vec{j}^* + rc^* = 0 \quad \text{in } \mathbb{R}^2, \quad t^* > 0,$$

or

$$c^*_{t^*} - \nabla\cdot(D\nabla c^*) + \nabla\cdot(\vec{u}c^*) + rc^* = 0, \quad t^* > 0.$$

<u>Alternative</u>: Fix $x_0 \in \mathbb{R}^2$, take $\Omega = B_r(x_0) = \{|x - x_0| < r\}$ then

$$(1) \Leftrightarrow \frac{1}{4\pi r_0^2} \iint_{B_r(x_0)} (c^*_{t^*} + \nabla \cdot \vec{j}^* + rc^*)\, dxdy = 0$$

and send $r_0 \to 0$ using the mean value theorem for integrals (or Taylor expansion of c^*, j^* about x_0).

<u>Initial condition:</u> At $t^* = 0$, all the ash is consetrated at $(x^*, y^*) = (0, 0)$, so the consentration has a point mass or delta function at $(0, 0)$:

$$c^*(x^*, y^*, 0) = C\delta(x^*)\delta(y^*)$$

where C is such that

$$N_0 = \iint_{\mathbb{R}^2} c^*(x, y, 0)\, dxdy = \iint C\delta(x)\delta(y)\, dxdy = C.$$

Problem 5

a) $v(\rho) = 1 - \rho$ and $j(\rho) = \rho v(\rho)$. Now we solve

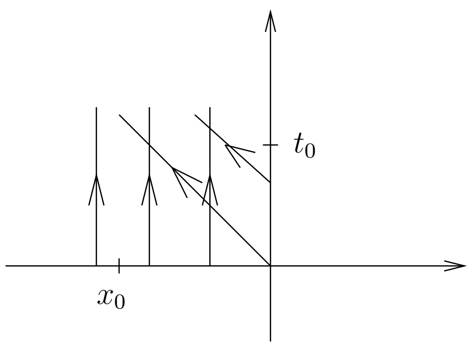
$$\begin{cases} \rho_t + (1 - 2\rho)\rho_x & = 0 \qquad x < 0,\ t > 0, \\ \rho & = \begin{cases} \frac{1}{2} & x < 0,\ t = 0, \\ \frac{1}{2} + \frac{\sqrt{2}}{4} & x = 0,\ t > 0, \end{cases} \end{cases}$$

The method of characteristics ($z(t) = \rho(x(t), t)$) gives

$$\begin{cases} \dot{x} = (1 - 2z) & x(t_0) = x_0 \\ \dot{z} = 0 & z(t_0) = \rho(x(t_0), t_0) = \begin{cases} \frac{1}{2} & t_0 = 0 \\ \frac{1}{2} + \frac{\sqrt{2}}{4} & x_0 = 0. \end{cases} \end{cases}$$

Solutions:

$$x = x_0 + (t - t_0)(1 - 2\rho(x_0, t_0)) = \begin{cases} x_0 & t_0 = 0 \\ (t - t_0)\left(-\frac{\sqrt{2}}{2}\right) & x_0 = 0. \end{cases}$$



Characteristics collide $\Rightarrow$ Shock solution:

$$\rho(x,t) = \begin{cases} \frac{1}{2}, & x < S(t) \\ \frac{1}{2} + \frac{\sqrt{2}}{4}, & x \geq S(t) \end{cases}$$

where the shock curve $S(t)$ satisfy Rankine-Hugoniot

$$\begin{aligned} \dot{S}(t) &= \frac{j(\rho(S^-(t)) - j(\rho(S^+(t))}{\rho(S^-(t)) - \rho(S^+(t))} \\ &= \frac{\frac{1}{2}\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right)\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)}{\frac{1}{2} - \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right)} \\ &= -\frac{\sqrt{2}}{4} \end{aligned}$$

and $S(0) = 0$, hence

$$S(t) = -\frac{\sqrt{2}}{4}t$$

b) $j_1(\rho) = \rho(1-\rho)$ and $j_3(\rho) = \rho(1-\rho)$, while

$$j_1(0,t) + j_2 = j_3(0,t), \tag{2}$$

is conservation of mass in the junction (what flows in must flow out). Since $\max j_3 = \frac{1}{4}$ (max capacity on Road 3), the combined traffic on Road 1 and 2 is too big for all $t > 0$. Therefore we will have max flux on Road 3 for all times:

$$j_3(0,t) = \frac{1}{4} \quad \underset{(5)}{\Rightarrow} \quad j_1(0,t) = \frac{1}{4} - j_2 = \frac{1}{8}.$$

c) The car speed is $v = 1 - \rho$, and $x = -1$ belongs to Road 1. From b) $j_1(0,t) = \frac{1}{8}$ and we must solve

$$\begin{cases} \rho_t + (1-2\rho)\rho_x = 0 & x < 0, t > 0 \\ j_1(\rho) = \frac{1}{8} & x = 0, t > 0 \\ \rho = \frac{1}{2} & x < 0, t = 0. \end{cases}$$

We convert from flux condition to a Dirichlet condition (a condition on ρ):

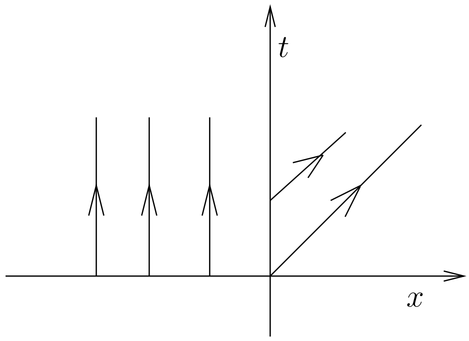
$$\frac{1}{8} = j_1(\rho) = \rho(1-\rho) \Rightarrow \rho = \frac{1}{2} \pm \frac{\sqrt{2}}{4} =: \rho_\pm.$$

<u>Case I</u>: $\rho = \rho_-$ at $x = 0$.

Characteristics (see a)): $\quad x = x_0 + (t - t_0)(1 - 2\rho(x_0, t_0)), \quad z = \rho(x_0, t_0)$.

Characteristics from $t = 0$: $z = \frac{1}{2} \quad \Rightarrow x = x_0$.

Characteristics from $x = 0$: $z = \rho_- \quad \Rightarrow x = (t - t_0)(1 - 2\rho_-) = \frac{\sqrt{2}}{2}(t - t_0)$.



Characteristics leave the domain $\{x < 0\}$ and we can not impose $\rho = \rho_-$ at $x = 0$. Moreover, flux continuity (conservation of cars) implies that $\frac{1}{4} = \lim j(\rho(0^-, t)) = j(0, t) \neq \frac{1}{8}$, a contradiction.

Case II = a) and is ok – in this case we have a inflow boundary and the flux condition is satisfied. By a),

$$\begin{aligned} v(-1,t) &= 1 - \rho(-1,t) \\ &= \begin{cases} 1 - \rho_+ & , -\frac{\sqrt{2}}{4}t < -1 \\ 1 - \frac{1}{2} & , -1 < -\frac{\sqrt{2}}{4}t \end{cases} \\ &= \begin{cases} \frac{1}{2} - \frac{\sqrt{2}}{4} & , t > 2\sqrt{2} \\ \frac{1}{2} & , 0 < t < 2\sqrt{2}. \end{cases} \end{aligned}$$