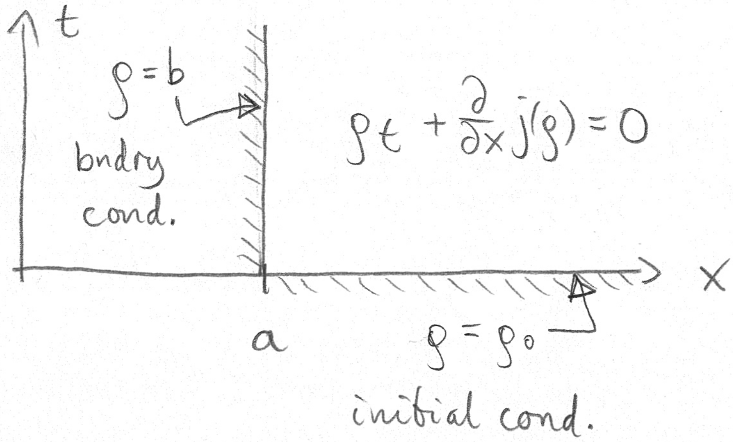
Matmod 13.10.11

A. Boundary value problem

$$(1)\quad \begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0 & x \in [a,\infty),\ t > 0 \\ \rho(a,t) = b & x = a,\quad t > 0 \\ \rho(x,0) = \rho_0(x) & x \in [a,\infty),\ t = 0 \end{cases}$$



Fact: Char.'s can start on bndry!

Char. eq'ns: $z(t) = \rho(x(t),t)$

$$\dot{x} = j'(z), \quad \dot{z} = 0$$

Init./cond.'ns:

a) along $t = 0$, $x \geq a$

$x(0) = x_0$, $\qquad x_0 \geq a$

$z(0) = \rho(x(0),0) = \rho_0(x_0)$

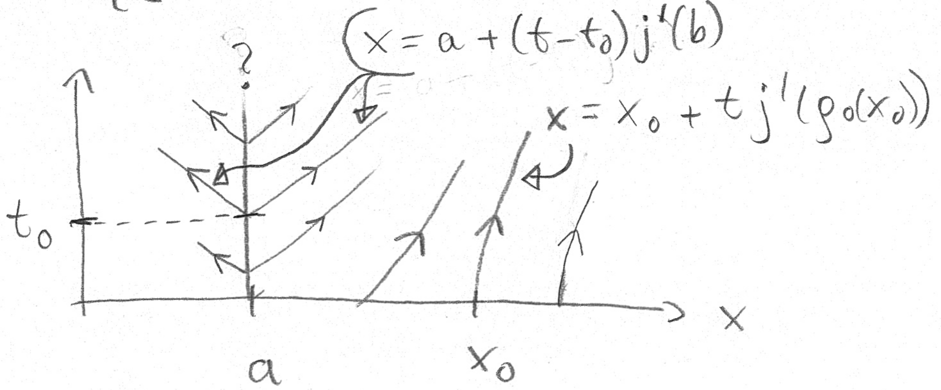
b) along $x = a$, $t > 0$:

$$x(t_0) = a \qquad t_0 > 0$$
$$z(t_0) = \rho(x(t_0), t_0) = \rho(a, t_0) = b$$

Sol'ns:

$$\begin{cases} x = x_0 + t\, j'(\rho_0(x_0)) \\ z = \rho_0(x_0) \end{cases} \qquad x_0 \geq a,\ t \geq 0$$

$$\begin{cases} x = a + (t - t_0) \cdot j'(b) \\ z = b \end{cases} \qquad t \geq t_0$$



<u>2 possibilities</u>:

1.) <u>Inflow</u>: $j'(b) \geq 0$

Char.'s at $x = a$ go <u>into</u> domain

$\Rightarrow$ Sol'ns found as before,

meth. of char.'s + shocks + rarefaction

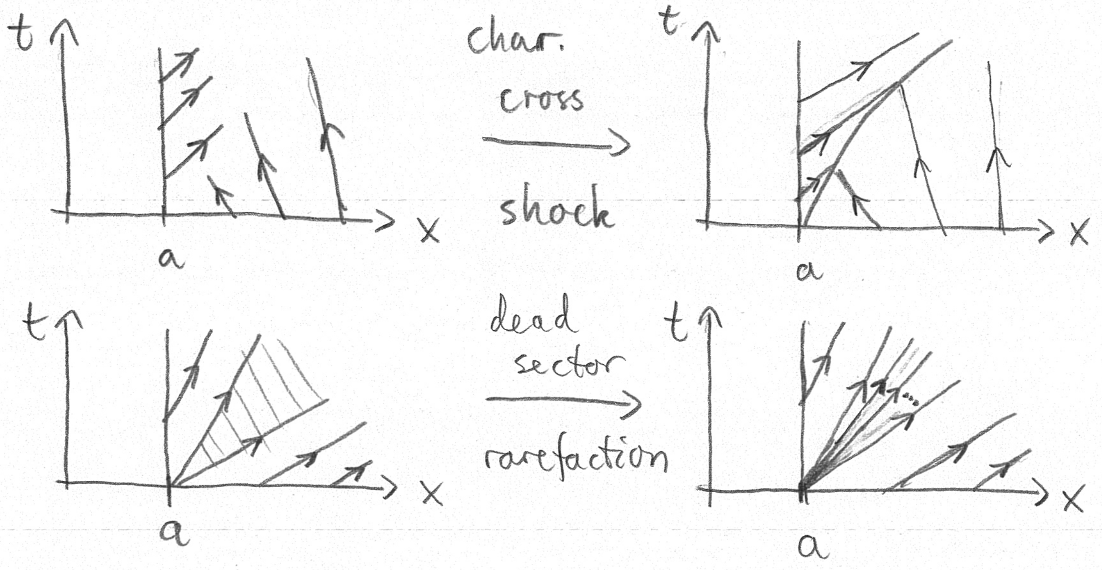
2.) <u>Outflow</u>: $j'(b) \leq 0$

Char.'s at $x = a$ <u>leave</u> domain

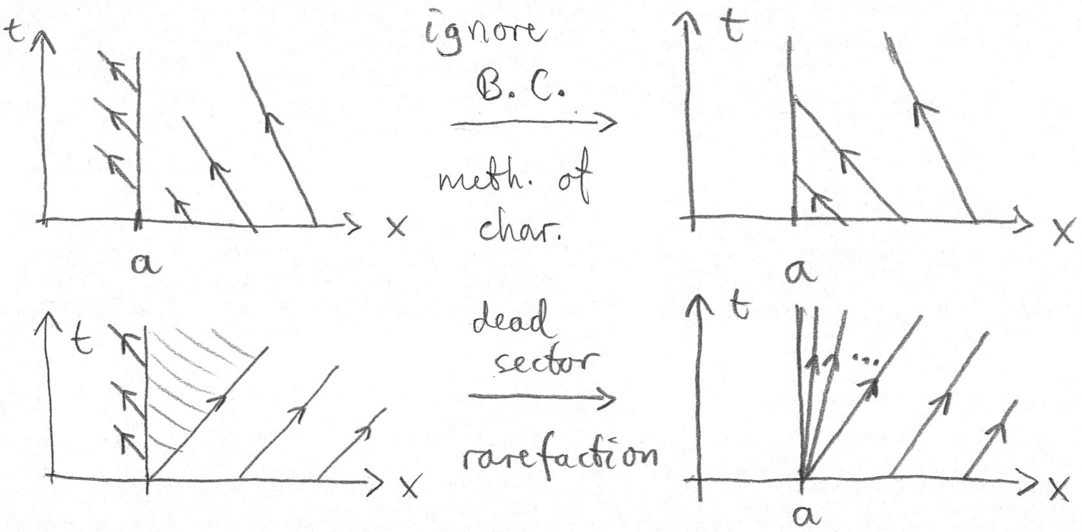
$\Longrightarrow$ sol'n does <u>not see</u> bndry cond. at $x = a$,

meth. of char. + shocks + rarefaction + ignore B.C.

Ex. 1: Inflow



Ex. 2: Outflow



Obs:

- Bndry. cond'ns (B.C.s) can <u>only</u> be imposed at inflow bndry's
- In Ex. 2, typically $\rho(a^+, t) \neq b$!

B. Flux - condition

$$(2)\quad \begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0 & x \geq a, \; t > 0 \\ j(\rho) = b & x = a, \; t > 0 \\ \rho(x,0) = \rho_0(x) & x \geq a, \; t = 0 \end{cases}$$

Obs: $j = 0$ at $x = a$

$\Rightarrow$ no flux (transport) across $x = a$

Idea: Convert (2) to (1) by solving

$$(3)\qquad j(\rho) = b \quad \text{for } \rho.$$

Obs: If (3) has many sol'ns, select

only sol'ns giving inflow bndry. (why?)

C. Traffic: Red light

Density of cars ρ:

$$\rho_t + \frac{\partial}{\partial x} j(\rho) = 0$$

where $j(\rho) = \rho(1-\rho)$.

Traffic light at $x = 0$:

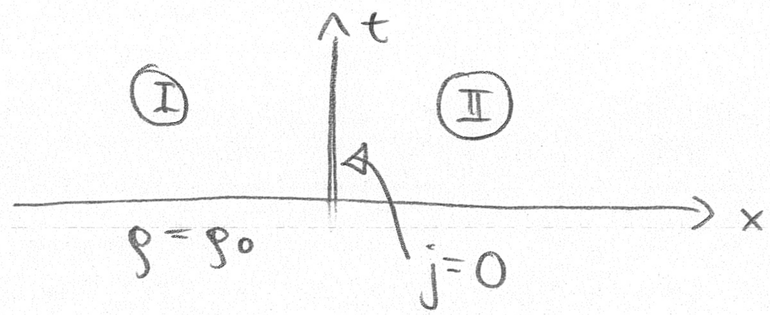
$$\begin{cases} \text{green for } t \leq 0 \\ \text{red for } t > 0 \end{cases}$$

Unif. traffic for $t \leq 0$:

$$\rho(x,t) \equiv \frac{1}{2}$$

Sol'n for $t > 0$:

Red light $\Rightarrow$ no flux cond'n at $x=0$!



$\rightsquigarrow$ 2 bndry value prob'ms, one in I ($x \leq 0$) and one in II ($x \geq 0$)

Bndry cond'ns:

$$j(\rho) = \rho(1-\rho) = 0 \Rightarrow \rho = 0 \text{ or } \rho = 1$$

Choose inflow values:

$$j'(\rho) = 1 - 2\rho = \begin{cases} 1 & , \rho = 0 \\ -1 & , \rho = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \rho = 1 \ (\rho = 0) \text{ gives inflow (outflow) in I} \\ \rho = 0 \ (\rho = 1) \text{ ——— " ——— in II} \end{cases}$$

(reasonable choices!)

Sol'n in I:

$$\begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0 & x < 0,\ t > 0 \\ \rho = 1 & x = 0,\ t > 0 \\ \rho = \frac{1}{2} & x < 0,\ t = 0 \end{cases}$$

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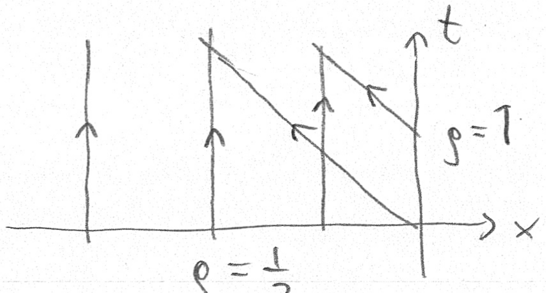
Meth. of char's

Init. char's: $x = x_0 + t \underbrace{j'(\tfrac{1}{2})}_{=0} = x_0 \quad ; \; x_0 \leq 0$

bndry-char's $\quad x = 0 + (t - t_0) \underbrace{j'(1)}_{=-1} = t_0 - t \; , \; t > t_0$



Char's collide $\Rightarrow$ shock

$$\dot{s} = \frac{\overset{= \frac{1}{4}}{j(\tfrac{1}{2})} - \overset{=0}{j(1)}}{\frac{1}{2} - 1} = -\frac{1}{2} \, , \; s(0) = 0$$

chk.

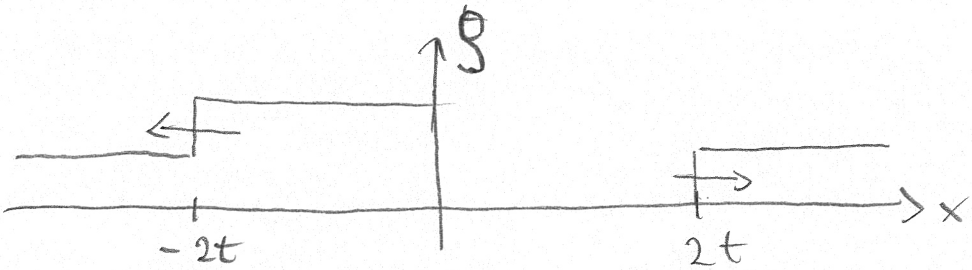
$$\Rightarrow \rho(x,t) = \begin{cases} 1 \, , & -\frac{1}{2}t < x < 0 \\ \frac{1}{2} \, , & x < -\frac{1}{2}t \end{cases}$$

Sol'n on II:

$$\begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0 \, , & x > 0, t > 0 \\ \rho = 0 \, , & x = 0, t > 0 \\ \rho = \frac{1}{2} \, , & x > 0, t = 0 \end{cases}$$

chk!

$$\rightsquigarrow \rho(x,t) = \begin{cases} 0 \, , & 0 \leq x \leq \frac{1}{2}t \\ 1 \, , & \frac{1}{2}t \leq x \end{cases}$$



D. Traffic: 2 → 1 lanes

2 → 1 lanes at $x = 0$

$$\Rightarrow v(\rho) = \begin{cases} 1 - \rho & x < 0 \\ 1 - \frac{\rho}{\frac{1}{2}} & x > 0 \end{cases}$$

max capacity in 1 lane road

$$\Rightarrow j(\rho) = \rho v(\rho) = \begin{cases} \rho(1-\rho) = j_-(\rho) & x < 0 \\ \rho(1-2\rho) = j_+(\rho) & x > 0 \end{cases}$$

Cond'n at $x = 0$: Flux continuity

$$j_-(\rho(0^-, t)) = j_+(\rho(0^+, t)) \quad t > 0$$

Conservation:
[If not: cars would be created/destroyed at $x = 0$]
(follows from cons. law in int form)

Obs: j is <u>discont.</u> in ρ but <u>cont.</u> in x!

($\Rightarrow \rho$ is discont. at $x = 0$)

Morning rush reach $x = 0$ at $t = 0$

$$\rho(x, 0) = \begin{cases} \frac{1}{2}, & x < 0 \\ 0, & x > 0 \end{cases}$$

Obs: $j_-(\rho(0^-, 0)) = \frac{1}{4} > \max\limits_{\rho} j_+(\rho) = \frac{1}{8}$

What is flux in $x > 0$?

Assume $j_+(\rho(0^+, t)) = \text{maximal} = \frac{1}{8}$

$\Rightarrow$ flux cond. $j = \frac{1}{8}$ at $x = 0$

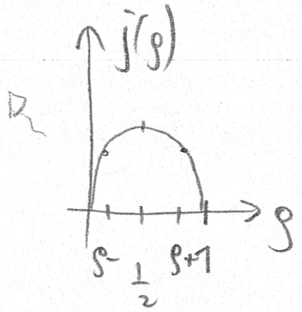
$$\Longrightarrow \begin{cases} x<0: & j_- = \frac{1}{8} \Rightarrow \rho_\pm = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{2}}\right) \\ x>0: & j_+ = \frac{1}{8} \Rightarrow \rho = \frac{1}{4} \end{cases}$$

Obs: $\rho_- < \frac{1}{2} < \rho_+$

- $j_-'(\rho_-) > 0$, $j_-'(\rho_+) < 0$

outflow inflow

- $j_+'(\frac{1}{4}) = 0$

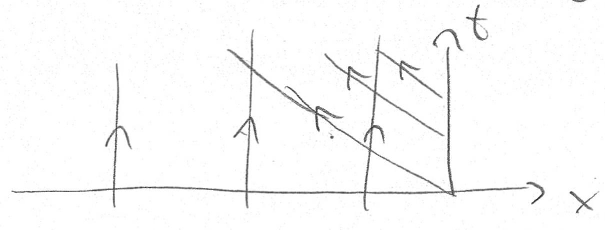


Sol'n for $x<0$:

$$\begin{cases} \rho_t + \frac{\partial}{\partial x} j_-(\rho) = 0 & x<0,\ t>0 \\ \rho = \rho_+ & x=0,\ t>0 \\ \rho = \frac{1}{2} & x<0,\ t=0 \end{cases}$$

Meth. of char's: $z(t) = \rho(x(t), t)$

$$x = \begin{cases} x_0 + t \underbrace{j_-'(\tfrac{1}{2})}_{=0} & x_0 \le 0,\ t \ge 0 \\ 0 + (t-t_0)\underbrace{j_-'(\rho_+)}_{<0} & t \ge t_0 \end{cases}$$



Char. collide $\Rightarrow$ shock

$$\dot{s} = \frac{j_-(\frac{1}{2}) - j_-(\rho_+)}{\frac{1}{2} - \rho_+} =: v < 0, \quad s(0) = 0$$

$$\Rightarrow \rho(x,t) = \begin{cases} \rho_+, & s(t) \le x \\ \frac{1}{2}, & x \le s(t) \end{cases}$$

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