Math 651 Mathematical Modeling and Simulation

Supplementary Notes, Part 1:

Dimensional Analysis, Scaling, and Regular Perturbation

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School of Mathematical and Statistical Sciences,
Hawassa University, Ethiopia

These notes have been prepared for the MSc Programme ***MASTMO** –**Mathematical and Statistical Modeling*** at School of Mathematical and Statistical Sciences, Hawassa University, Ethiopia

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Contents

1 DIMENSIONAL ANALYSIS

1.1 The Basis of Dimensional Analysis

Dimensional Analysis is a technique based on two simple axioms about nature:

- *All relations between physical quantities must be dimensionally correct*
- *No physical relation should depend on any particular set of units*

Even if these axioms sound trivial and obvious, they lead to a powerful, simple and quite useful tool in mathematical modelling. At the same time, it illustrates how important it is to uncover the mathematical essence in two general and apparently vague statements. Let us now investigate the axioms in somewhat more detail. A physical quantity has

- *dimension*
- *unit*
- *numerical value*

Dimension means, e.g. *Length*, *Time*, *Mass*, or combinations. The dimension of a physical quantity is given once and for all and will never change. When we work with physical quantities we need units, and as we know, there is a huge amount of units. For a length we have cm, m, km, foot, inch etc. When the unit changes, the numerical value of the quantity also changes, as illustrated in Fig. 1.

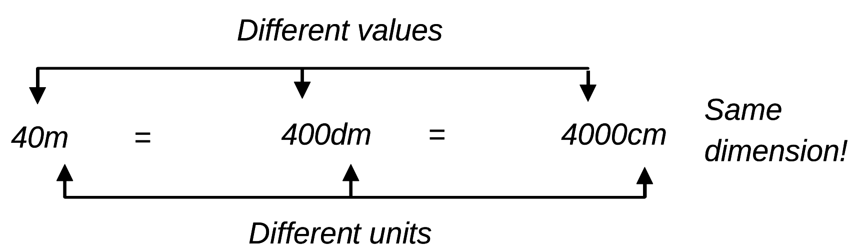

Figure 1: The numerical value changes when the unit changes, but the physical dimension remains the same.

Let R be a physical quantity. The reader should have observed by now that we have not really defined what a physical quantity is. *Wikipedia* defines a physical quantity as a physical property that can be quantified in terms of numbers. Thus, the mass of Earth is a physical quantity whereas a dice showing a 6 for a rock concert is not. It is convenient to have a notation for the unit of R, and we shall write this as $[R]$. The value of R when we use a certain unit is denoted $v(R)$. This is not a standard notation, but convenient for the moment. Hence, R has a unit and a numerical value,

$$R = v(R)\,[R]\,.$$

Quantity	Dimension symbol	SI unit
Mass	M	kilo (kg)
Length	L	meter (m)
Time	T	second (s)
Electric current	I	ampere (A)
Absolute temperature	Θ	Kelvin (K)
amount of substance	N	Mole (mol)
luminous intensity	J	Candela (Cd)

Table 1: Fundamental physical units in the SI-system. Exact definitions may be found in Wikipedia.

That a physical relation or equation is *dimensionally correct* means we do not add apples and pears, or that each side of an equality do not have different units. It is obvious that the well-known formula $S = \frac{1}{2}gt^2$ is dimensionally correct since

$$\begin{aligned} [S] &= \mathrm{m}, \\ [g] &= \mathrm{m/s}^2, \\ [t] &= \mathrm{s}. \end{aligned} \tag{1}$$

On the contrary, you often find, even in textbooks, equations of the form

$$S = 4.9 \times t^2. \tag{2}$$

In order to be meaningful t must be measured in seconds and s in metres, and it will not work for other pairs of units. Such relations should not be used. If you can not state your equations in a dimensionally correct form, you have probably not a clear idea of what is going on. In physics there is a set of dimensions forming so-to-speak the *atoms*. None of these depend on the others and the dimensions along with corresponding units in the *International System of Units* (abbreviated the *SI-system*) are listed in Table 1. All physical quantities have units which are power combinations of the basic SI-system units (this may be considered as the definition of a physical quantity).

1.2 Buckingham's Pi-theorem

Buckingham's Pi-theorem extracts the mathematical content of the two axioms in the introduction. The somewhat strange name comes from the dimensionless variables that we end up with when we apply the theorem. These are often referred to as π_1, π_2, $\cdots$ (Π is also used instead of π).

Let us look at what *a relationship between physical quantities* means. A relationship is a relation or a formula, that is, an equation that we can write

$$\Phi(R_1, R_2, \cdots, R_M) = 0, \tag{3}$$

where Φ is a certain function. We could also write this relation in other ways, *e.g.*

$$R_1 = \Psi(R_2, \cdots, R_M). \tag{4}$$

If we choose a set of fundamental units, and uses these in a consistent way for all the involved variables, we should now have

$$\Phi(v(R_1), v(R_2), \cdots, v(R_M)) = 0.$$

In practice we could, for example, specify $v(R_2), \cdots, v(R_M)$, and then calculate $v(R_1)$ from the relationship. However, it may well happen that there is no valid relationship between $v(R1), \cdots, v(R_M)$. If we consider S and t where $[S] = \mathrm{m}$ and $[t] = \mathrm{s}$, there is no valid physical relation that contains *only* these two variables. We need at least one more quantity, as in the well-known relations $S = gt^2/2$ or $S = Vt$. By simply looking at the quantities and their units, it is possible to decide whether they at all can be combined into a sensible relation.

To investigate this further, it is smart to create a so-called *dimension matrix* containing the exponents of the fundamental units in the units for the quantities we have. If $[S] = \mathrm{m}$, $[t] = \mathrm{s}$ and $[g] = \mathrm{ms}^2$, the dimension matrix will be

	S	t	g
m	1	0	1
s	0	1	-2

. (5)

Let us already here point out that we use the familiar units in the first column. Actually, it would be more correct to use universal dimension assignments such as L for *length*, T for *time*, M for *mass*, etc. This is used in many textbooks. The dimension matrix surveys what the dimensions of involved variables are.

Let now, in general, $F_1, F_2, \cdots, F_N$ denote the fundamental units in Table 1. The units of any physical quantity may be expressed by means of these, for example, the unit for *energy* is "$\mathrm{kgm}^2/\mathrm{s}^2$". Generally, we may thus write

$$\begin{aligned} [R_1] &= F_1^{a_{11}} F_2^{a_{21}} \cdots F_N^{a_{N1}}, \\ &\vdots \\ [R_M] &= F_1^{a_{1M}} F_2^{a_{2M}} \cdots F_N^{a_{NM}}. \end{aligned} \tag{6}$$

This gives the dimension matrix $\mathbf{A}$:

$$\begin{array}{|c|ccccc|} \hline & R_1 & R_2 & & \cdots & R_M \\ \hline F_1 & a_{11} & & & \cdots & a_{1M} \\ F_2 & \vdots & & \mathbf{A} & & \vdots \\ \vdots & & & & \ddots & \\ F_N & a_{N1} & & \cdots & & a_{NM} \\ \hline \end{array} . \tag{7}$$

We say that $R_1, \cdots, R_r$ have *independent dimension* if it is impossible to make a (non-trivial) dimensionless combination of $R_1, \cdots, R_r$ of the form

$$R_1^{\lambda_1} R_2^{\lambda_2} \times \cdots \times R_r^{\lambda_r}. \tag{8}$$

In order to see what this means, we determine the unit of this expression:

$$\begin{aligned} \left[R_1^{\lambda_1} R_2^{\lambda_2} \times \cdots \times R_r^{\lambda_r}\right] = F_1^{a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots a_{1r}\lambda_r} \times \\ \times F_2^{a_{21}\lambda_1 + a_{22}\lambda_2 + \cdots a_{2r}\lambda_r} \times \cdots \\ \cdots \times F_N^{a_{N1}\lambda_1 + a_{N2}\lambda_2 + \cdots a_{Nr}\lambda_r}. \end{aligned} \tag{9}$$

That $R_1, \cdots, R_r$ have independent dimensions means that the equation system

$$\begin{aligned} a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots a_{1r}\lambda_1 &= 0, \\ a_{21}\lambda_1 + a_{22}\lambda_2 + \cdots a_{2r}\lambda_r &= 0, \\ &\vdots \\ a_{N1}\lambda_1 + a_{N2}\lambda_2 + \cdots a_{Nr}\lambda_r &= 0, \end{aligned} \tag{10}$$

only has the trivial solution

$$\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0. \tag{11}$$

We may recall from the theory of linear equations that this will happen if and only if the matrix columns,

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{N1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{N2} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1r} \\ \vdots \\ a_{Nr} \end{bmatrix}, \tag{12}$$

are *linearly independent*. Thus, we have proved that $R_1, \cdots, R_r$ have independent dimensions if and only if the corresponding dimensional matrix has linearly independent columns. Since the vectors have unit N, we must have $r \leq N$ for this to be possible. Going back to the example above, we see that S and t have independent dimensions. The same applies for $\{S, g\}$ and $\{t, g\}$. On the contrary, S, t and g do not have independent dimensions.

If we now have a general dimension matrix, the maximum number of variables with independent dimension is equal to the maximum number of columns that are linearly independent. This is known in linear algebra as the *rank* of the matrix. Let us assume that we have organized ourselves so that $R_1, \cdots, R_r$ have independent dimension, and that r is the rank of A, rank(A). We may assume that $r < M$ (If $r = M$, all quantities have independent dimension and there will be no non-trivial physical relationship between them). From this assumption, we may use $R_1, \cdots, R_r$ to form combinations involving $R_{r+1}, \cdots, R_M$ such that

$$\begin{aligned} \pi_1 &= R_{r+1}/(R_1^\bullet \times \cdots \times R_r^\bullet), \\ \pi_2 &= R_{r+2}/(R_1^\bullet \times \cdots \times R_r^\bullet), \\ &\vdots \\ \pi_{M-r} &= R_M/(R_1^\bullet \times \cdots \times R_r^\bullet), \end{aligned} \tag{13}$$

is dimensionless. Here "$^\bullet$" means suitable exponents so as to make the π-s dimensionless. If we then have a relation

$$\Phi(R_1, R_2, \cdots, R_M) = 0, \tag{14}$$

it is possible to replace $R_{r+1}, \cdots, R_M$ and arrange the expression such that we end with a new, but equivalent relation,

$$\Psi(R_1, R_2, \cdots, R_r, \pi_1, \cdots, \pi_{M-r}) = 0. \tag{15}$$

We now claim: *If $R_1, \cdots, R_r$ have independent dimensions, it is possible to choose a set of fundamental units such that the values $v(R_1), v(R_2), \cdots, v(R_r)$ become arbitrarily specified positive numbers*!

This is easy to see if we have just one quantity, *e.g.* $R_1 = 40$m. If we measure R_1 in centimeters, $v(R_1) = 4000$, while measured in kilometers,$v(R_1) = 0.04$, and so on.

If the claim is correct, we have obtained the following interesting situation. Whatever units we decide to use,

$$\Psi\left(v(R_1),v(R_2),\cdots,v(R_r),v(\pi_1),\cdots,v(\pi_{M-r})\right)=0. \tag{16}$$

While the first r variables may take any positive value depending on how we choose the units, the latest $M-r$ variables remain constant during this change. As a function of M variables Ψ will therefore always be completely unaffected by the values of the r first arguments. In other words, Ψ can really only depend on $\pi_1,\cdots,\pi_{M-r}$! We have thus reduced our relation with M variables, $\Phi(R_1,R_2,\cdots,R_M)=0$ to a new relation $\Psi(\pi_1,\cdots,\pi_{M-r})=0$ with only $M-r$ variables. Apart from the claim above, which is proved below, we have now proven

Buckingham's Pi-teorem:

If there exists a (physically proper) relation

$$\Phi(R_1,R_2,\cdots,R_M)=0 \tag{17}$$

between the quantities $R_1,R_2,\cdots,R_M$, there also exists an equivalent relation

$$\Psi(\pi_1,\cdots,\pi_{M-r})=0, \tag{18}$$

where r is the rank of the dimension matrix.

Note that the theorem *assumes* that there is a relationship between $R_1,R_2,\cdots$, and R_M. This has to be ensured, or at least assumed, before we apply the theorem. In fact, Buckingham's Pi-teorem may also prove that no such relation exists. Buckingham Pi-theorem reduces the number of parameters, and if none of the dimensionless π-s are completely redundant, $M-r$ is also the least possible number of variables we have in our problem.

In the proof above we applied $R_1,R_2,\cdots,R_r$ to create the dimensionless combinations. These variables are often called *core variables.* Usually there are several possibilities for the core variables, and what is appropriate depends on the problem.

It is easy to set up a formal procedure to determine the π-s (see Sec. 1.3.2 below). This is also found in the textbooks, *e.g.* [4], but most of the time it is just as easy to find the combinations by a simple inspection. Note that the number of dimensionless variables will be the same regardless the choice of core variables and combinations. In principle, all are equally valid.

In the remainder of this section, we shall, for those particularly interested, show the above claim.

We assume therefore that $R_1,\cdots,R_r$ has independent dimension, and that dimension matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{Nr} \end{bmatrix} \tag{19}$$

has rank r. A well-known proposition from linear algebra say that the *rank of a matrix* $\mathbf{A}$ *is the same as the rank of the transposed matrix* $\mathbf{A}^T$. It turns out that this sentence is exactly what we need to complete the proof.

Let us assume that, using the fundamental units $F_1,F_2,\cdots,F_N$, R_i has the values $v_F(R_i)$, $i=1,\cdots,r$. For another set of fundamental units, $G_1,G_2,\cdots,G_N$, the numerical values will be

$v_G(R_i),\ i = 1, \cdots, r$. Let $x_i = F_i/G_i$. Then,

$$\begin{aligned} R_i &= v_F(R_i) F_1^{a_{1i}} F_2^{a_{2i}} \cdots F_N^{a_{Ni}} \\ &= v_F(R_i) x_1^{a_{1i}} G_1^{a_{1i}} x_2^{a_{2i}} G_2^{a_{2i}} \cdots x_N^{a_{Ni}} G_N^{a_{Ni}} \\ &= v_F(R_i) x_1^{a_{1i}} x_2^{a_{2i}} \cdots x_N^{a_{Ni}} G_1^{a_{1i}} G_2^{a_{2i}} \cdots G_N^{a_{Ni}} \\ &= v_G(R_i) G_1^{a_{1i}} G_2^{a_{2i}} \cdots G_N^{a_{Ni}}. \end{aligned} \tag{20}$$

Thus,

$$v_F(R_i) x_1^{a_{1i}} x_2^{a_{2i}} \cdots x_N^{a_{Ni}} = v_G(R_i),\ i = 1, \cdots, r. \tag{21}$$

If we take the logarithm of both sides, we end up with a linear system of equations of the form

$$\begin{aligned} a_{11} \log(x_1) + \cdots + a_{N1} \log(x_N) &= \log(v_G(R_1)) - \log(v_F(R_1)), \\ a_{12} \log(x_1) + \cdots + a_{N2} \log(x_N) &= \log(v_G(R_2)) - \log(v_F(R_2)), \\ &\cdots \\ a_{1r} \log(x_1) + \cdots + a_{Nr} \log(x_N) &= \log(v_G(R_r)) - \log(v_F(R_r)). \end{aligned} \tag{22}$$

We recognize the coefficient matrix in this equation system as the transposed of the dimension matrix. Since we have r equations and, according to the proposition from linear algebra, r linearly independent columns in the coefficient matrix, this system will have solutions $\log(x_1), \cdots, \log(x_N)$ (not necessarily unique) *regardless* of the choice of the right-hand side. But this means precisely that we may first specify $v_G(R_1), \cdots, v_G(R_r)$ to whatever we want, then find $\log(x_1), \cdots, \log(x_N)$, then $x_1, \cdots, x_N$, and finally select custom G-units , $G_i = F_i/x_i$, $i = 1, \cdots, r$.

1.3 Some Applications of Dimensional Analysis

1.3.1 The First Atomic Bomb Explosion

The following example of the use of dimensional analysis has become a classic. The English physicist G. I. Taylor watched an amateur film of the first American atomic bomb explosion in the Nevada desert, and measured the radius (r) of the fireball as a function of time (t), see Fig. 2. He argued that r, apart from t, should depend on the energy (E) that is released in the explosion and the density (ρ) of the air, since the flame-front needs to accelerate the mass of the surrounding air. Thus, he assumed that we have a relation $\Phi(r, t, \rho, e) = 0$, and set up the following dimension matrix

	r	t	ρ	e
kg	0	0	1	1
m	1	0	-3	2
s	0	1	0	-2

(23)

Note that we can find the unit of energy, expressed in terms of the fundamental units, by consulting well-known formulas from physics. For example, we know that energy is force×distance, and that force is mass×acceleration. This gives us

$$[e] = \text{Nm} = (\text{kgm/s}^2)\text{m} = \text{kgm}^2/\text{s}^2.$$

The dimension matrix above has rank 3, and, according to Buckingham there is $4 - 3 = 1$ dimensionless parameter. Since the equation is simply $\Psi(\pi_1) = 0$, we assume it has a unique solution such that we may write

$$\pi_1 = C, \tag{24}$$

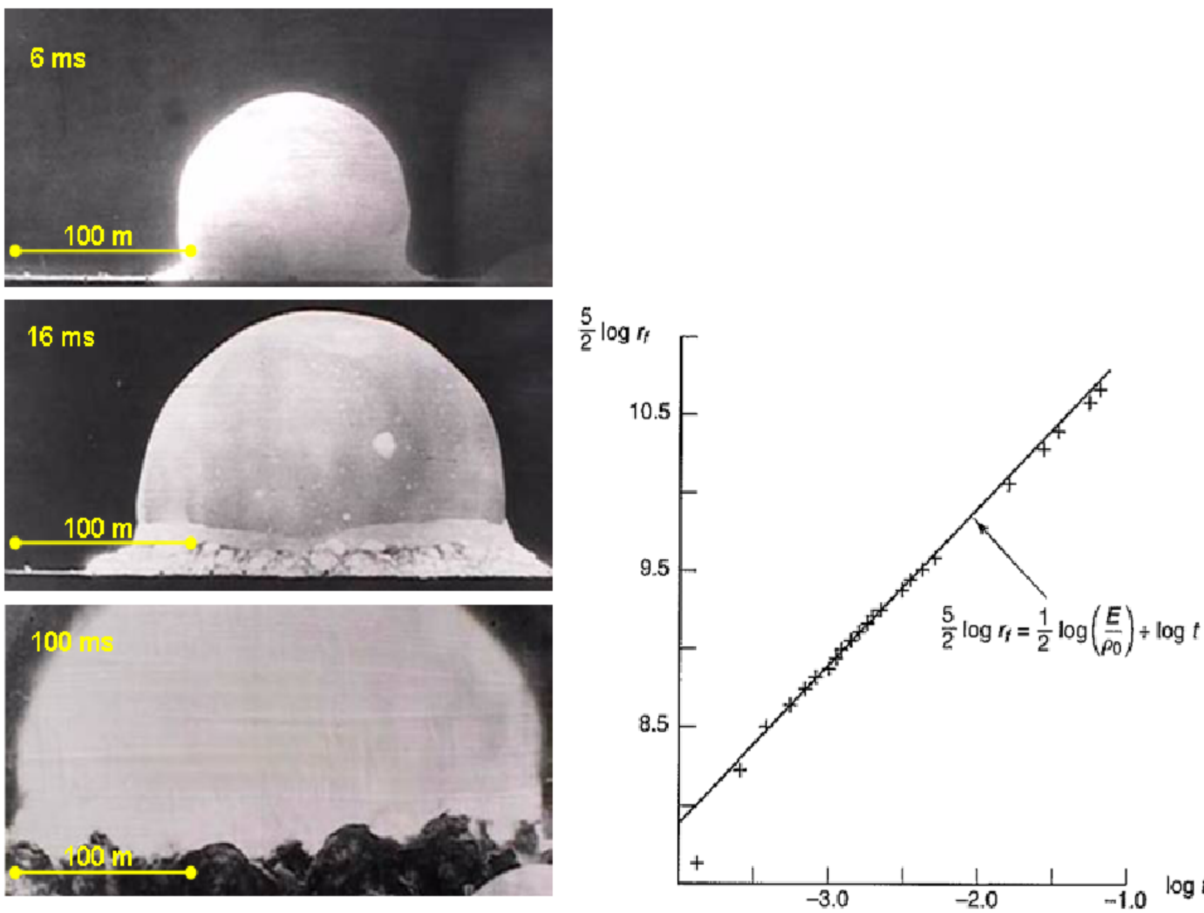

Figure 2: This series of images from the first atomic bomb explosjon shows a fireball growing with time. To the right is shown a copy of the figure G.I. Taylor published based on the whole sequence of images (See the book of Barenblatt, s. 47 – 50).

where C is an unknown constant. We may find π_1 by *trial and error*. First,

$$\left[\frac{e}{\rho}\right] = \frac{\text{Nm}}{\text{kg/m}^3} = \frac{\text{kgm}^2}{\text{s}^2}\frac{\text{m}^3}{\text{kg}} = \frac{\text{m}^5}{\text{s}^2}, \tag{25}$$

and then we observe that the following combination is dimensionless

$$\pi_1 = \frac{e}{\rho}\frac{t^2}{r^5}. \tag{26}$$

This gives us the following simple formula:

$$e = C\rho\frac{r^5}{t^2}. \tag{27}$$

We are not able to determine the constant C, but from an amateur film G.I. Taylor was able to find the ratio r^5/t^2, and by assuming $C = 1$ (best guess!), he got that the released energy $e \approx 10^{14}$J. It turned out that this was within a factor of 2 of the correct. The publication of the energy, which of course was "top secret", caused great confusion among the Americans when this was published as a letter in The Times.

1.3.2 A General Recipe for Finding Dimensionless Combinations

If it is difficult to see the dimensionless combinations directly, it is possible to put up a system of equations for the exponents. The method may be illustrated using the example above. Here we are looking for a dimensionless combination of the form $\pi = e^x\rho^y r^z t^u$ and must therefore determine $\{x, y, z, u\}$. Since we already know the units of the variables,

$$\begin{aligned}[\pi] &= [e^x\rho^y r^z t^u] \\ &= \left(\text{kg}^x\text{m}^{2x}\text{s}^{-2x}\right)\left(\text{kg}^y\text{m}^{-3y}\right)\left(\text{m}^z\right)\left(\text{s}^u\right) \\ &= \text{kg}^{x+y}\text{m}^{2x-3y+z}\text{s}^{-2x+u}.\end{aligned}$$

Here π should be dimensionless, and therefore,

$$\begin{aligned} x + y &= 0, \\ 2x - 3y + z &= 0, \\ -2x + u &= 0. \end{aligned}$$

There is no unique solution, but we choose $x = 1$, it follows easily that $y = -1$, $z = -5$, and $u = 2$, in other words, exactly what we already knew. There is no reason to use this cumbersome method if it is possible to see the result directly.

1.3.3 Pythagoras' Theorem

Since a right-angled triangle is completely determined by the length of the hypotenuse (c) and the smallest angle (α_{min}), there must be a relationship between surface area (A) of the triangle, the length of the hypotenuse and the angle,

$$\Phi(c, A, \alpha_{\text{min}}) = 0. \tag{28}$$

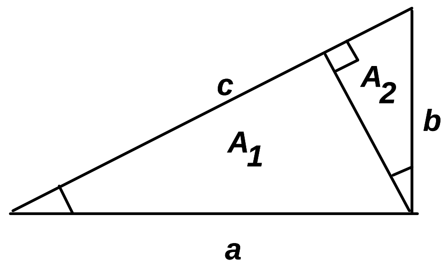

Figure 3: The area of the big triangle is the sum of the equally shaped smaller triangles

It is easy to set up the dimension matrix:

	A	c	$\alpha_{\min}$
m	2	1	0

(An angle is measured in radians, which is a ratio between two lengths and thus dimensionless). Since the rank of the matrix is 1, there are two dimensionless parameters, A/c^2 and $\alpha_{\min}$, and we end up with a relation $A = c^2 f(\alpha_{\min})$. From Fig. 3 it is obvious that $A = A_1 + A_2$ for the areas, and therefore $c^2 f(\alpha_{\min}) = a^2 f(\alpha_{\min}) + b^2 f(\alpha_{\min})$, or $c^2 = a^2 + b^2$, which is yet another proof of this famous result (apparently not yet listed among the 93 other proofs found at [6]).

1.3.4 Fluid Flows in Tubes

This example, giving an expression for the friction factor to use for fluids flowing in tubes, is quite famous and very useful in engineering. It requires some background in fluid mechanics.

We shall find an expression for the pressure drop in a cylindrical tube which contains a flowing fluid, and we assume that the variables in Table 2 are important.

Concerning *viscosity*, μ, this is a proportionality constant between *shear stress* (*force/area unit*), for example σ_{yx}, and changes in the speed per. unit length, $\partial u/\partial y$, normal to the force direction (consult a textbook in fluid mechanics or *Internet* for more information). For a so-called *Newtonian fluid* (like water and air), $\sigma_{yx} = \mu \partial u/\partial y$. Thus, the unit for μ is:

$$[\mu] = \frac{[\sigma_{yx}]}{[\partial u/\partial y]} = \frac{(\mathrm{kgm/s^2})/\mathrm{m}^2}{(\mathrm{m/s})/\mathrm{m}} = \frac{\mathrm{kg}}{\mathrm{ms}}. \tag{29}$$

The measure of the wall roughness (e) could, *e.g.* be the typical standard deviation around the mean (think of a cement tube with rough walls). Clearly, the size of e may vary over several orders of magnitude from very smooth glass tubes, to steel pipes, cement tubes or hydropower tunnels in rocks! From Table 2 we may derive the dimension matrix in Table 3.

We see that the rank is 3 (the maximum it can be), and consequently, we have $7 - 3 = 4$ dimensionless quantities. In this example it is not very smart to use P as a core variable because we want to express P in terms of the other variables. One possible choice for core variables is $\{V, D, \rho\}$ since

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ -1 & 0 & 0 \end{vmatrix} \neq 0, \tag{30}$$

Quantity	Name	Unit
Pressure	P	N/m^2=kg/s^2m
Mean fluid velocity	V	m/s
Tube diameter	D	m
Tube length	L	m
wall roughness	e	m
Viscosity	μ	kg/ms
Density of the fluid	ρ	kg/m^3

Table 2: Quantities that may be included in the expression for the pressure drop in the pipe.

	P	V	D	L	e	μ	ρ
kg	1	0	0	0	0	1	1
m	−1	1	1	1	1	−1	−3
s	−2	−1	0	0	0	−1	0

Table 3: Dimension matrix for the variables in the expression for the pressure loss.

and the columns are thus linearly independent.

The next step is to form dimensionless combinations where the remaining variables are included. It is easy to check that the following combinations are possible choices:

$$\begin{aligned} \pi_1 &= P/(v^2\rho), \\ \pi_2 &= L/D, \\ \pi_3 &= e/D, \\ \pi_4 &= vD\rho/\mu. \end{aligned} \tag{31}$$

(There are other possibilities, but since we are not the first ones to carry out this exercise, we show only the most useful one). Since we want P to be expressed by the other variables, it is reasonable to think of a relationshipof the form

$$\pi_1 = \Phi(\pi_2, \pi_3, \pi_4). \tag{32}$$

Now it is also reasonable to assume (and this is verified by experiments) that the pressure drop is proportional to the tube L. Hence, it should be possible to write

$$\pi_1 = \pi_2\Phi_2(\pi_3, \pi_4), \tag{33}$$

or

$$P = \frac{L\rho V^2}{D}\Phi_2(\frac{e}{D}, \frac{VD\rho}{\mu}). \tag{34}$$

In fluid mechanics it is common to replace Φ_2 with $2f_F$, where f_F is called *Fanning's friction factor,*

$$f_F = f_F(\frac{e}{D}, \frac{VD\rho}{\mu}). \tag{35}$$

The combination e/D is denoted by ε and is known as the tube's *relative roughness.* The second expression,

$$Re = \frac{\rho Dv}{\mu}, \tag{36}$$

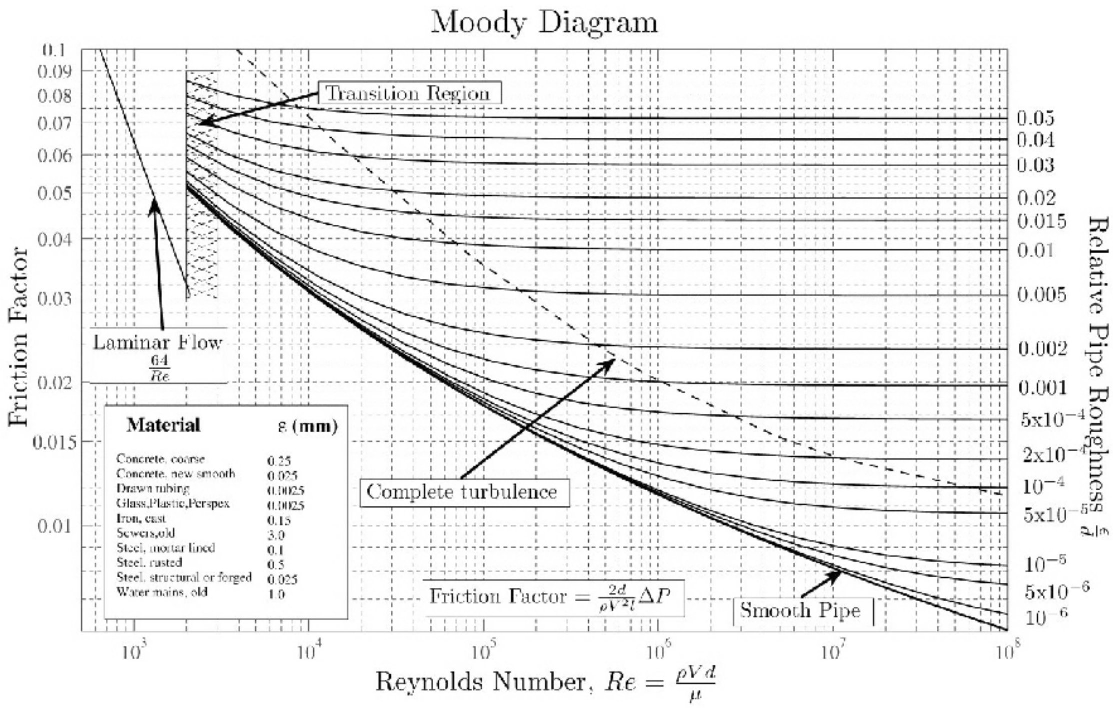

Figure 4: Moody-diagram copied from the *Wikipedia Common* image data base.

is the famous *Reynolds Number.* Just by applying dimensional analysis we have established that

$$P = 2\frac{L\rho V^2}{D} f_F(\varepsilon, Re). \tag{37}$$

(In the literature one will also encounter the friction factor $f_D = 4f_F$ called *Darcy's friction factor*). The friction factor f_F must be determined from more advanced theory and experiments, and in 1944 L.W. Moody presented his the famous diagram which is now called a *Moody diagram*, see Fig. 4. The diagram together with interactive code for calculating f_F are also found in numerous versions on *Internet*.

For those specially interested, one can mention that for very low Reynolds numbers ($Re < 2000$) the flow will be *laminar*. This is the so-called Hagen-Poiseuille flow with a parabolic velocity profile over a smooth tube with circular cross section. For such flow one can show analytically that $P = 32L\mu V/D^2$, *i.e.* $f_F = 16/Re$. For the rest of the chart, there are more or less empirical expressions available. When $Re > 6000$ we have

$$f_F = \max(f_C, f_N), \tag{38}$$

where f_C is the solution of *Colebrook's equation*

$$\frac{1}{f_C^{1/2}} = -1.74 \log\left(\frac{\varepsilon}{3.7} + \frac{1.25}{Re \times f_C^{1/2}}\right), \tag{39}$$

and f_N is given from *Nikuradse's relation* for fully developed turbulent flow,

$$f_N = \frac{4}{(1.14 - 0.87 \log \varepsilon)^2}. \tag{40}$$

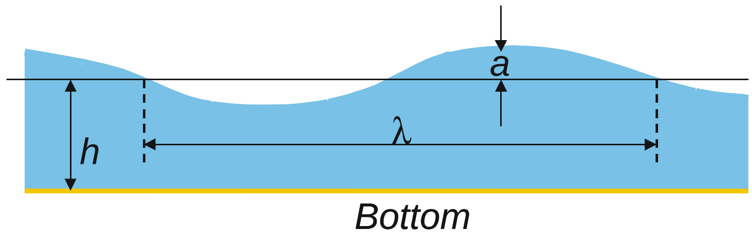

Figure 5: A regular wave on the surface of water.

In the transition between laminar and turbulent flow, the flow is unstable and can switch in an unpredictable way between being laminar or turbulent.

1.3.5 Water Waves

In one space dimension, we can write a regular wave on water surface

$$\eta(x,t) = a\cos(kx - \omega t), \tag{41}$$

where a is the wave amplitude, x the space coordinate, t is time, $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength, $\omega = 2\pi/T$ is the angular frequency and T the wave period, see Fig. 5.

For water waves, k and ω can not be chosen arbitrarily, but must satisfy a *dispersion relation*

$$\omega^2 = f(k, h, a, \cdots). \tag{42}$$

It is reasonable that the angular frequency ω occurs with a second power. Positive and negative frequencies corresponding to waves that move to the right and left, respectively (for positive k). Waves on water may be generated by the wind, boats etc. and are maintained by gravity. For very short waves, $\lambda = O(1\text{cm})$, the surface tension keeps the wave going. The surface tension (or stress), σ is characterized by a surface tension coefficient T (not to be confused with the period) that connects the surface curvature and the tension. In one dimension, the expression is $\sigma = T\partial^2\eta/\partial x^2$. The unit for T is thus

$$[T] = \frac{[\sigma]}{[\partial^2\eta/\partial x^2]} = \left(\frac{\text{kgm}}{\text{s}^2}\frac{1}{\text{m}^2}\right)\frac{\text{m}^2}{\text{m}} = \frac{\text{kg}}{\text{s}^2}. \tag{43}$$

Since gravity is important, the gravitational acceleration g and the water density ρ are also possible parameters in the dispersion relation. We neglect the effect of air motion over the waves. Thus, we end up with the following assumption about the dispersion relation:

$$\omega^2 = f(k, a, h, g, \rho, T). \tag{44}$$

The corresponding dimension matrix is displayed in Table 4. We easily see that the matrix has rank 3, and consequently, there are $7 - 3 = 4$ dimensionless combinations. Of several possibilities we choose $\{k, g, \rho\}$ as our core variables. It is not particularly smart to use ω^2, since we do not want ω^2 to enter on the right side of the equation. Furthermore, either ρ or T need to be

	ω^2	k	a	h	g	ρ	T
m	0	−1	1	1	1	−3	0
s	−2	0	0	0	−2	0	−2
kg	0	0	0	0	0	1	1

Table 4: Dimension matrix for the dispersion relation.

involved since these are the only variables that contain kg in their units. It is now easy to find four dimensionless combinations,

$$\begin{aligned}\pi_1 &= \frac{\omega^2}{kg},\\ \pi_2 &= ak,\\ \pi_3 &= hk,\\ \pi_4 &= \frac{Tk^2}{\rho g},\end{aligned} \tag{45}$$

and we find

$$\omega^2 = gk\Phi(ak, hk, \frac{Tk^2}{\rho g}). \tag{46}$$

In this formula there are several special cases:

- The wave has very small amplitude compared to the wavelength, $ak \ll 1$
- The water depth is large relative to wavelength, $hk \gg 1$
- The wavelength is much larger than 1cm, $\frac{Tk^2}{\rho g} \ll 1$ (follows from the numerical value of T)

If all three conditions are present, we could write $\omega^2 \approx gk\Phi(0, \infty, 0)$. A more refined analysis (by solving the differential equations for water waves) shows that the relation in this case is $\omega^2 \approx gk$ and that $\Phi(0, \infty, 0) = 1$. If the depth is not so large, we obtain $\omega^2 = gk\Phi(0, hk, 0)$, and a closer analysis here shows that

$$\omega^2 = gk \tanh(hk). \tag{47}$$

If the depth is large, and we have very short waves, only surface tension and not gravity is of importance. A simplified dimensional matrix could then be

	ω^2	k	T
m	0	−1	0
s	−2	0	−2
kg	0	0	1

, (48)

but since only T depends on "kg", it is impossible to combine T with the two others and form a dimensionless combination. Thus, we need another parameter to match "kg" , and the only possibility is ρ. We leave to the reader to show that this gives us

$$\omega^2 = C\frac{Tk^3}{\rho}. \tag{49}$$

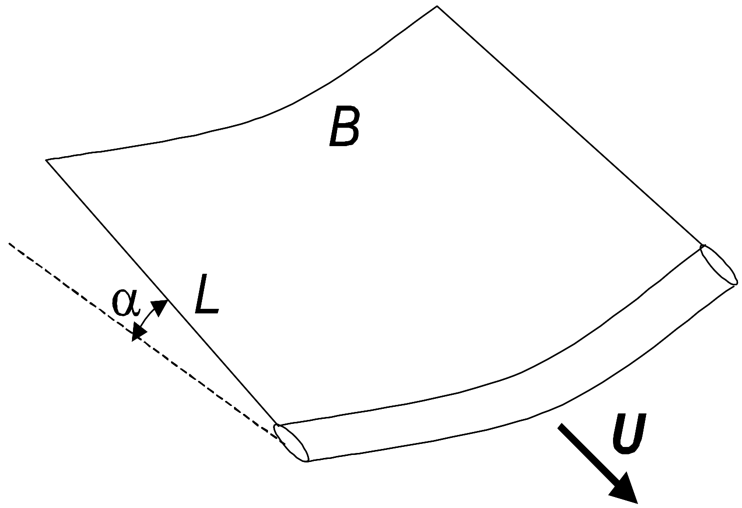

Figure 6: Sketch of the paper airplane in the text.

It turns out that also in this case, $C = 1$. In general, it is possible to show by analytical methods that for $ak \ll 1$, we will have

$$\omega^2 = gk \tanh(kh) \left(1 + \frac{Tk^2}{\rho g}\right). \tag{50}$$

1.3.6 Design of Paper Airplanes

What is the optimal shape of a paper airplane? Even if we restrict ourselves to one kind of models, this is not a simple question and we expect to do a lot of experimentation. Before we start, it may be smart to carry out some dimensional analysis. We shall focus on models that have performed well in *Scientific American*'s paper airplane competitions. The airplanes are made by taking a sheet of paper of length L_0 and width B and fold it with small folds from one side until the center of gravity lies approximately 1/4 from the folded edge, as shown in Fig. 6.

After the folding, the plane has length L. Instead of folding, it is also possible, using a little stiffer paper, to position the center of gravity correctly by using one or more clips in front of the sheet. Ideally, such a wing should slide with constant velocity U in a fixed angle α with the horizontal plane. Assume that one task is to investigate how the speed depends on the length, width, and weight of the airplane. The paper's weight per unit is denoted ρ_p (kg/m^2), the air density ρ_a (kg/m^3), and air viscosity ν (m^2/s).

Let us first consider the friction force F between the airplane and the air. This force must in any case depend on the size and speed of the plane, i.e., L, B and U. Furthermore, we expect that the viscosity of the air is of importance. If we look at L, B, U and ν, we find that none of the units of these variables include kg, and since this occurs in force, we need a few more parameters. It is reasonable to choose ρ_a, while there is no reason why ρ_p should enter the expression for the force. There are other parameters that can be expected to have little impact on the friction force, such as how smooth the paper is, how thick is the fold, etc., but we should have listed the most important ones.

The dimension matrix is constructed similarly to above

$$\begin{array}{|c|cccccc|}\hline & F & B & L & U & \nu & \rho_a \\ \hline \text{kg} & 1 & 0 & 0 & 0 & 0 & 1 \\ \text{m} & 1 & 1 & 1 & 1 & 2 & 0 \\ \text{s} & -2 & 0 & 0 & -1 & -1 & -3 \\ \hline \end{array}. \tag{51}$$

We leave to the reader to show that this gives us three dimensionless variables which can be arranged so that

$$F = L^2 U^2 \rho_a \Psi\left(\frac{L}{B}, \frac{LU}{\nu}\right). \tag{52}$$

Alternatively, one may write

$$F = LBU^2 \rho_a \tilde{\Psi}\left(\frac{L}{B}, \frac{LU}{\nu}\right), \tag{53}$$

and when $B \gg L$, it would be reasonable to replace $\tilde{\Psi}\left(\frac{L}{B}, \frac{LU}{\nu}\right)$ with a function of only one unknown, i.e.

$$\Phi\left(\frac{LU}{\nu}\right) = \tilde{\Psi}\left(0, \frac{LU}{\nu}\right). \tag{54}$$

If the aircraft moves with a constant speed in a fixed angle with the horizontal, the friction force has to balance gravity. Then

$$F = Mg \sin\alpha = \rho_p (L_0 B) \sin\alpha. \tag{55}$$

(In addition, there must be sufficient lift for the airplane to stay in the air). For a wide plane, the speed may then be expressed as

$$U^2 = \frac{\rho_p}{\rho_a} \frac{L_0}{L} g \sin\alpha \Phi\left(\frac{LU}{\nu}\right). \tag{56}$$

The combination LU/ν is again the Reynolds number, and as we see, simple dimensional analysis has given us much insight which we can take with us further in the investigation.

1.4 Summary

In this chapter we have seen, based on two fairly obvious axioms about the nature, that it is possible to derive the quite powerful Buckingham pi-theorem. These axioms are basically laws of nature, of our universe. The theorem is easy to use, but requires that there really is a relationship between the quantities we have listed. In practice, this can be problematic to determine.

If we were asked to find the eigenfrequency ω of a mathematical pendulum, we would assume that this depends on the length of the pendulum's rod (L), the gravitational acceleration (g), the pendulum's position angle from the vertical at the start (α), and the mass of the bob (m). Based on these quantities, there should exist a relationship

$$\Phi(\omega_0, L, g, \alpha, m) = 0. \tag{57}$$

Here we observe, however, that this is impossible, since the mass is the only quantity containing kg in its unit. Either we must remove m, or there must be an additional quantity that we have forgotten. Since is seems impossible to find other reasonable parameters to include, we are forced to remove m. This leads to the following useful observation:

- *Each fundamental unit must occur in at least two of the quantities.*

The standard procedure now gives us

$$\omega_0 = (g/L)^{1/2} f(\alpha). \tag{58}$$

The results of the dimensional analysis are not unambiguous. Instead of writing

$$\Psi(\pi_1, \pi_2, \cdots, \pi_{M-r}) = 0, \tag{59}$$

we could just as well write

$$\begin{aligned} \pi_1 &= f(\pi_2, \cdots, \pi_{M-r}), \\ f(\pi_1, \pi_2) &= g(\pi_3, \cdots, \pi_{M-r}), \\ &\vdots \end{aligned} \tag{60}$$

Here, we use what is appropriate. There is no reason to say that one way of writing the formula is more correct than another. In addition, the dimensionless combinations are not unique. If π_1 is dimensionless, then so is also $\sqrt{\pi_1},\ 1/\pi_1,\ \pi_1^2$. With more experience, one will often recognize common combinations such as Reynolds number, etc.

The core variables was the subset that we used to form the dimensionless combinations. Usually, there are also several possibilities here. If we are interested in finding how a variable (such as R_1) depends on the others, it is reasonable to avoid using R_1 as one of the core variables. In that way, we find a relation of the form $R_1 = \phi(R_2, R_3, \cdots, R_M)$, that is, R_1 does not enter into the arguments in ϕ.

We have treated dimensional analysis as a method to simplify the relationships between physical quantities. Dimensional analysis is used to obtain an overview and can indicate whether we really understand what we are doing.

One of the best properties of dimension analysis is that it gives us a formulation containing the minimum number of free variables. This is in particular valuable for experimental work in the lab or at the computer.

If we decide to find the frequency of a mathematical pendulum by means of experiments only, and assume that $\omega_0 = \Phi(L, g, \alpha, m)$, we may have to determine the function Φ by selecting 10 different values for each variable, that is, perform a total of 10^4 experiments. If we first use dimensional analysis, we realize that it is enough to use only one pendulum, vary the angle α for a reasonable set of values,and then plot α against $\omega_0\,(L/g)^{1/2}$ in order to determine the function $f\,(\alpha)$ in the expression from the dimensional analysis,

$$\omega_0 = \sqrt{\frac{g}{L}} f\,(\alpha)\,.$$

A similar simplification is also important to do in order to save the number of numerical experiments on a computer, and before setting up experimental plans in statistical experiments.

Dimensional analysis is also crucial when working with *scale models*, that is, doing experiments with models scaled down (or up) in size. Ideally, one would like that the dimensionless combinations are the same for the model as for the original (this is called the *scale laws*).

All comprehensive textbooks on mechanics describes dimensional analysis. For example, both [4] and [3] has nice introductions, while [2] is considered a classic. Moreover, the *Internet* has several million references to *Dimensional Analysis*.

2 SCALING

2.1 Introducing Scaled Variables

After establishing a mathematical model in the form of an equation, it will be necessary to introduce dimensionless variables. Usually it is not difficult to do this, but it can be carried out in several ways, and it is not always easy to see what is most appropriate way. However, there exists an intelligent way of doing this called *scaling the equations*. When the equations are scaled, it is easy to see which parts are important and which are less important. It can be difficult to scale equations, and in any case this will depend on the problem we are considering, even if the equation is basically the same all the time. Somewhat simplified we can say that the scales force us to think about the situation, and in this way we gain insight into what we are doing. The theory in this chapter is mainly taken from [3].

- *To scale a variable u^* means to write the variable as*

$$u^* = Uu, \tag{61}$$

  where $[U] = [u^*]$, U is of the same order of magnitude as u^*, and u is of order 1.

Here U is the characteristic size of u^*: If we use U as our unit of measurement, u is neither particularly large nor particularly small. This is a somewhat imprecise definition, but it reflects the fact that scales are not always very well defined.

Until getting used to scaling variables, it is handy to have a notation in order to distinguish between the original variables with units, and the new dimensionless variables. We will do this, as suggested in [3], by attaching * on the original variables, and remove * after the variable has been made dimensionless. After a while, we become tired of writing *, and understand the transition from the context.

Let us consider a variable u^* which is a function of time t^*. It is usually reasonable to use

$$U = \max_{t^*} |u^*(t^*)|$$

as a scale for u^*, even if the minimum of u^* is much smaller. Then, at least $|u| \leq 1$. In practice, this often means to *estimate* the maximum value, since we may not know u^* in detail.

It will also be necessary to find scales for time. Sometimes the maximum value of t^*may be used, but more often the scale is defined as a period over which u^* *varies significantly*. If $u^*(t^*) = \sin at^*$, a reasonable time scale would be $1/a$, since u^* then varies from 0 to 1. As suggested in [3], it is often possible to find a reasonable time scale by looking at (or estimate)

$$\frac{\max|u^*(t^*)|}{\max|du^*/dt^*|}. \tag{62}$$

Such expressions must be used with common sense, and when working with scales, we are not very careful about extra factors such as 2, π, etc. Scaling is not an exact science, – often a rough estimate is all we need.

- *To scale an equation means to introduce dimensionless variables based on the scales of the variables in the equation.*

Depending on the situation we are in, the same equation could be scaled in several ways. After the equation is scaled, it will be clear what are important and less important parts of the equation (if not all are equally important). Often one will be able to get approximate solutions by solving the equation when the less important parts are removed. Knowing the scales of the variables of a mathematical model requires knowledge and physical understanding, and is one of the most important things we do in mathematical modeling. As will be seen below, scaling is not nearly as easy as it sounds. A good example is one of the main modeling examples in [3], where the authors, several years after the book was published, discovered that the time scale they had suggested was not really appropriate (Quite recently, the authors of these notes have suggested a completely different scaling of the same equations).

2.2 Order of Magnitude

We say that the function $f(x)$ is of the *order of magnitude* $g(x)$ when $x \to a$ if there exist two finite numbers $\{m, M\}$, $0 < m < 1 < M$, such that

$$m \leq \frac{f(x)}{g(x)} \leq M \tag{63}$$

for $x \to a$. This is written

$$f(x) = \mathcal{O}(g(x)), \; x \to a, \tag{64}$$

and expressed in words as "*$f(x)$ is of order $g(x)$ when x is close to a*". Some like to require that $m = \sqrt{1/10}$ and $M = \sqrt{10}$ (what is then $\log_{10} m$ and $\log_{10} M$?), but we prefer a more informal use, *e.g.*

$$\log(1+x) - x = \mathcal{O}\left(x^2\right) \text{ for small } x\text{-s.} \tag{65}$$

For series, the first non-zero term is called the *leading order* term, *i.e.* $4x^3 + 3x^4 + 5x^5 + \cdots$ is *of leading order* x^3 for small x-s.

A slightly different symbol, "$o()$" is more precise: We write

$$f(x) = o(g(x)) \text{ when } x \to a, \tag{66}$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0. \tag{67}$$

Thus,

$$\sin x - x + x^3/6 = \mathcal{O}\left(x^5\right), \tag{68}$$

$$\sin x - x + x^3/6 = o\left(x^4\right), \tag{69}$$

when $x \to 0$.

2.3 A Simple Case Study

In the following artificial and simple example we shall see how scales change depending on the nature of the problem. The example is trivial and easy to solve analytically. The assumption about the friction force is not very realistic.

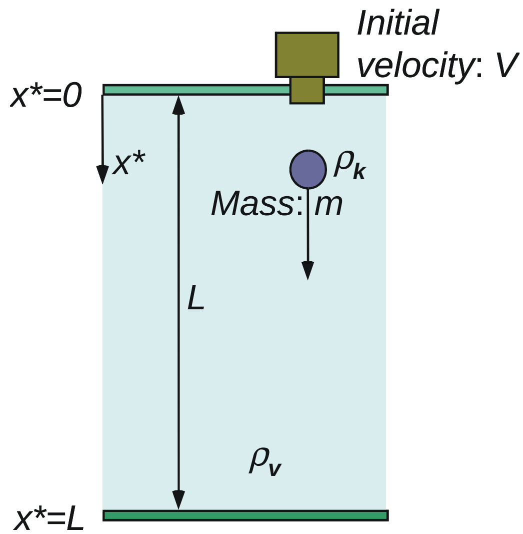

Figure 7: Ball falling or fired into a viscous fluid.

A spherical ball is fired vertically into a viscous fluid as illustrated in Fig. 7. The ball's initial speed is V and the forces acting on the ball is

$$\begin{array}{ll} \text{Gravity:} & gm \\ \text{Friction:} & -k\frac{dx^*}{dt^*} \\ \text{Buoyancy:} & -gm\frac{\rho_v}{\rho_k} \end{array} \tag{70}$$

(Here, ρ_v is the fluid density and ρ_k the density of the ball). The equation of motion follows from Newton's Law and we assume that the ball starts at $x^* = 0$ with velocity V:

$$m\frac{d^2x^*}{dt^{*2}} = gm - k\frac{dx^*}{dt^*} - mg\frac{\rho_v}{\rho_k}, \tag{71}$$

$$x^*(0) = 0, \; \frac{dx^*}{dt^*}(0) = V. \tag{72}$$

We shall also assume that $\rho_v < \rho_k$, such that the ball does not eventually float up to the surface, and we replace $g(1 - \frac{\rho_v}{\rho_k})$ with a modified g so that the problem simplifies to

$$m\frac{d^2x^*}{dt^{*2}} = gm - k\frac{dx^*}{dt^*},$$

$$x^*(0) = 0, \; \frac{dx^*}{dt^*}(0) = V. \tag{73}$$

In this case we can imagine a number of special cases. If the ball had fallen freely with zero initial velocity, it would, at $x = L$, have reached the speed v_{FF} where $v_{FF} = \sqrt{2Lg}$ (Vertical motion under constant acceleration). If, on the other hand, the medium is *very* viscous (think of syrup!), the ball will after a while fall with constant speed v_0 determined by

$$0 = gm - kv_0, \tag{74}$$

i.e. $v_0 = \frac{gm}{k}$.

Below we shall consider three different situations, and it will turn out that the ratio between v_{FF} and v_0 is crucial.

2.3.1 Case A: The friction is large – what happens initially is not very important

This is a situation where either L or the viscosity (here expressed by the constant k) is so great that the ball falls at a constant speed over most of the distance. Assuming that the ball has speed v_0 all the way, we may estimate the time it takes to go from $x^* = 0$ to $x^* = L$ to about

$$T_0 = \frac{L}{v_0} = \frac{Lk}{mg}, \tag{75}$$

and this gives us a reasonable time scale. Depending on the size of V, the actual time the ball uses could be slightly larger or smaller than T_0. There is however an implicit assumption here that V is not very large compared to v_0. The length scale is not a problem, we use L and introduce dimensionless variables x and t as

$$\begin{aligned} x^* &= Lx, \\ t^* &= \frac{Lk}{mg}t. \end{aligned} \tag{76}$$

By bringing this into the equations, we obtain

$$\begin{aligned} m\frac{d^2(Lx)}{d\left(\frac{Lk}{mg}t\right)^2} + k\frac{d(Lx)}{d\left(\frac{Lk}{mg}t\right)} &= gm, \\ Lx(0) = 0, \frac{d(Lx)}{d\left(\frac{Lk}{mg}t\right)}(0) &= V, \end{aligned} \tag{77}$$

and after simplification,

$$\begin{aligned} \frac{gm^2}{Lk^2}\frac{d^2x}{dt^2} + \frac{dx}{dt} &= 1, \\ x(0) = 0, \ \frac{dx}{dt}(0) &= \frac{V}{v_0}. \end{aligned} \tag{78}$$

In addition to the variables x and t, the problem contains two dimensionless parameters:

$$\begin{aligned} \varepsilon &= \frac{gm^2}{Lk^2}, \\ \mu &= V/v_0. \end{aligned} \tag{79}$$

We note that

$$\varepsilon = \frac{gm^2}{Lk^2} = 2\frac{1}{2Lg}\left(\frac{gm}{k}\right)^2 = 2\frac{v_0^2}{v_{FF}^2} = 2\left(\frac{v_0}{v_{FF}}\right)^2. \tag{80}$$

Thus, ε is a small parameter (compared to 1) if $v_0 \ll v_{FF}$. It is characteristic for this case that the speed v_0 is much less than the speed the ball would have had at $x^* = L$ if it fell freely. It is typical that when we have scaled the equations, the dimensionless parameters have interesting interpretations that we may apply for *hindsight*.

After the scaling is complete, the equation has the form

$$\begin{aligned} \varepsilon\frac{d^2x}{dt^2} + \frac{dx}{dt} = 1, \ x(0) = 0, \ \frac{dx}{dt}(0) = \mu, \\ \varepsilon = 2\frac{v_0^2}{v_{FF}^2}, \ \mu = \frac{V}{v_0}. \end{aligned}$$

As mentioned above, there is here an assumption that μ is not particularly large. In that case, one might imagine another time scale (see Case C below).

2.3.2 Case B: Small friction. The ball falls approximately freely. V is small compared to v_{FF}.

This problem could have been the same as in Case A, but now with the difference that L is so small that the ball never reaches speeds near v_0. Thus, friction is of little importance.

Again, L is a natural length scale for the x^*. If the ball fell freely and $V = 0$, the ball would fall with nearly constant acceleration, and the time it takes to fall to $x^* = L$ would roughly be $\sqrt{2L/g}$. Since we have already introduced v_{FF}, we apply $T_0 = L/v_{FF}$ as our scale. Certainly, T_0 is only about the half of $\sqrt{2L/g}$, but we do not care about this for a scale estimate. We have already assumed that the speed V is so small that it does not affect the time scale. With these deliberations, we may write

$$
\begin{aligned}
x^* &= Lx, \\
t^* &= \frac{L}{v_{FF}} t,
\end{aligned}
\tag{81}
$$

and obtain

$$
mL \frac{2Lg}{L^2} \frac{d^2x}{dt^2} + kL \frac{v_{FF}}{L} \frac{dx}{dt} = mg, \; x(0) = 0, \; \frac{dx}{dt}(0) = \frac{V}{v_{FF}}, \tag{82}
$$

and finally

$$
2\frac{d^2x}{d\bar{t}^2} + \varepsilon \frac{dx}{dt}(0) = 1, \; x(0) = 0, \; \frac{dx}{dt}(0) = \mu, \tag{83}
$$

$$
\varepsilon = \frac{v_{FF}}{v_0}, \; \mu = \frac{V}{v_{FF}}. \tag{84}
$$

Note that the definition of ε has changed compared to Case A, and here, ε is a small parameter if $v_{FF} \ll v_0$. This is a characteristic feature of Case B. The scaling above is only reasonable if V is small compared to v_{FF}. If V is greater than v_{FF}, but still smaller than v_0, the ratio L/V could be a reasonable time scale. We leave to the reader to complete the scaling in this case.

2.3.3 Case C: The ball is released into a highly viscous medium. The initial velocity V is much larger than v_0

In this case, we expect that friction dominates over gravity, and we estimate the length and time scales by looking at the approximate equation

$$
m\frac{d^2x^*}{dt^{*2}} = -kV, \; x^*(0) = 0, \; \frac{dx^*}{dt^*}(0) = V. \tag{85}
$$

If this had been the exact equation, the ball would stop for $t^* = T_0 = \frac{m}{k}$ (since $\frac{dx^*}{dt^*} = V - \frac{Vk}{m}t^*$). An associated length scale (where we again disregard a factor of 2) will then be

$$
L = VT_0 = \frac{Vm}{k}. \tag{86}
$$

Case	Characteristics	Length scale	Time scale	Equation	Parameters
A	$v_0 \ll v_{FF}$	L	L/v_0	$\varepsilon\ddot{x} + \dot{x} = 1$	$\varepsilon = 2\frac{v_0^2}{v_{FF}}$, $\mu = \frac{V}{v_0}$
B	$v_0 \gg v_{FF}, V < v_{FF}$	L	L/v_{FF}	$2\ddot{x} + \varepsilon\dot{x} = 1$	$\varepsilon = \frac{v_{FF}}{v_0}$, $\mu = v/v_{FF}$
C	$V \gg v_0$	mv/k	m/k	$\ddot{x} + \dot{x} = \varepsilon$	$\varepsilon = \frac{v_0}{v}$, $\mu = 1$

Table 5: A summary of the scaling example

We introduce $x^* = \frac{mV}{k}x$ and $t^* = \frac{m}{k}t$:

$$m\frac{mV}{k}\frac{k^2}{m^2}\frac{d^2x}{dt^2} + k\frac{mV}{k}\frac{k}{m}\frac{dx}{dt} = mg,\ \ x(0) = 0,\ \ \frac{dx}{dt}(0) = 1, \tag{87}$$

which, after some simplification becomes

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = \varepsilon,\ \ x(0) = 0,\ \ \frac{dx}{dt}(0) = 1, \tag{88}$$

$$\varepsilon = \frac{v_0}{V}. \tag{89}$$

In this situation, ε is a small parameter when $V \gg v_0$, and this is the characteristic feature for Case C.

2.3.4 Summary

We have now seen three different situations where weight has been put on various parts of the equation. The problem has, in addition to V, two characteristic speeds, namely $v_{FF} = \sqrt{2gL}$ and $v_0 = \frac{gm}{k}$, and the various situations above are characterized by the mutual size of these speeds. We summarize the results in Table 5.

In all three situations we end up with a parameter ε which is typically small. The related terms in the equation are also small, and by neglecting the terms of order ε, we obtain the simplified equations.

Although it is the rule rather than the exception that we end with terms of different size in a scaled equation, it is also possible that all terms happen to be of the same magnitude.

We leave to the reader to show that the exact solution is

$$x^*(t^*) = \frac{gm}{k}t^* + \left(V - \frac{gm}{k}\right)\frac{m}{k}\left(1 - e^{-t^*/(m/k)}\right), \tag{90}$$

and the graph in Fig. 8 shows how the exact solutions relate to the situations we have seen. Note that the cases we have considered by no means cover the entire chart. By setting $\varepsilon = 0$ for all three situations above, we obtain simplified equations, but Case A is special. With $\varepsilon = 0$, the equation and initial conditions become

$$\dot{x}_0 = 1, \tag{91}$$

$$x_0(0) = 0, \tag{92}$$

$$\dot{x}_0(0) = \mu, \tag{93}$$

and unless μ happens to be 1, it is *impossible* to solve the simplified problem exactly. The general solution to Eq. (91) is

$$x_0\left(t\right) = C + t, \tag{94}$$

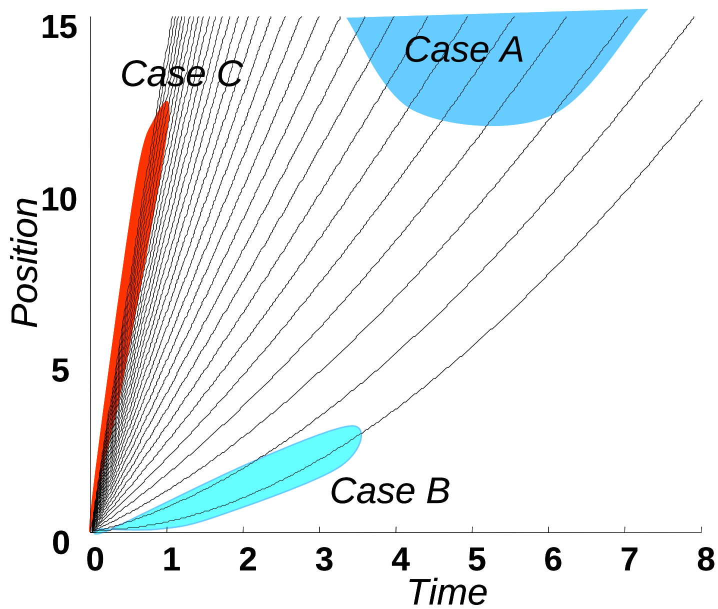

Figure 8: The figure shows the exact paths and an indication of the three situations we have considered.

and since the speed is 1 (v_0 in the original variables), the form is reasonable. The approximate solution is simply not valid near 0, and in order to determine the correct $C = C(\mu)$ a special technique (*singular perturbation*) is required.

In Case B, the approximate equation is

$$\begin{aligned} 2\ddot{x}_0 &= 1, \\ x_0(0) &= 0, \\ \dot{x}_0(0) &= \mu, \end{aligned} \tag{95}$$

which we immediately solve as

$$x_0(t) = t^2/4 + \mu t. \tag{96}$$

We can check the approximate solution by inserting it in the exact equation,

$$2\ddot{x}_0 + \varepsilon \dot{x}_0 - 1 = \varepsilon \left(\frac{t}{2} + \mu \right). \tag{97}$$

The error on the RHS increases with time, and this is reasonable since the approximate solution is not at all limited by friction.

The equation for Case C has approximate solution

$$x_0(t) = 1 - e^{-t}, \tag{98}$$

so here $x_0(t) < 1$ for all t.

Although the solutions in B and C obviously have their weakness, they are great for the situations they are supposed to cover. Convince yourself by drawing the approximate and exact solutions for some choice of ε and μ.

2.4 Scaling Considerations

Arguments based on scale considerations have proven to be quite useful in many contexts, but they require some physical insight and creativity, and are not always so easy to perform.

2.4.1 Turbulence

Fluids are mixed (on the microscopic level) by *molecular diffusion*, and (on the macroscopic level) by *convection*. Molecular diffusion is related to the kinematic viscosity of the fluid (ν, $[\nu] = \mathrm{m}^2/\mathrm{s}$), while convection is the macroscopic motion, typical by visible whirls, observed when we move the spoon around in a cup of tea, or when watching the whirling water in a river, for example.

Suppose we consider a whirl with diameter L. The time scale associated with L and convection with a velocity scale U will be

$$t_K = L/U. \tag{99}$$

The time scale associated with diffusion over a length L may likewise be expressed by the kinematic viscosity, ν and L. The only possibility is

$$t_D = \frac{L^2}{\nu}. \tag{100}$$

We observe that the quotient between these two scales is

$$\frac{t_D}{t_K} = \frac{L^2 U}{\nu\, L} = \frac{LU}{\nu} = Re, \tag{101}$$

which is a new meaning of the well-known *Reynolds number,* also mentioned above. A Reynolds number $Re \ll 1$ indicates that the mixing is dominated by molecular diffusion, whereas $Re \gg 1$ means that it is dominated by convection.

The value of ν for water is about $10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$. Consider a typical river with width $L = 100$m, and $U = 1$m / s. Then

$$Re \approx \frac{100 \cdot 1}{10^{-6}} = 10^8, \tag{102}$$

and the mixing of the water in the river is entirely dominated by convection.

In turbulent flow large vortices initiate motion of small vortices which, in turn, set into motion (and keep alive) even smaller vortices, and so on. In the very small vortices viscosity will reduce the motion and the kinetic energy is eventually transferred into heat. The kinetic energy dissipation (loss of energy) is mainly from these small vortices with a length scale l' and velocity scales u'. We can estimate the energy loss (E) per. time and unit mass by assuming that $E = E(l', u', \nu)$, and that $E \propto u'^2$ (in other words, proportional to the kinetic energy that is present in the smallest vortices). Simple dimensional analysis then gives

$$E \propto \nu \left(\frac{u'}{l'}\right)^2, \tag{103}$$

and an estimate for E will be $E = \nu \left(\frac{u'}{l'}\right)^2$. From the above we can further assume that the smallest vortices have $Re \approx 1$, or $t_K \approx t_D$, that is

$$\frac{l'u'}{\nu} = 1. \tag{104}$$

Thus,

$$l' = \left(\frac{\nu^3}{E}\right)^{1/4}, \tag{105}$$
$$u' = (\nu E)^{1/4}.$$

These scales are called the *Kolmogorov's micro scales* in turbulence theory. These are the smallest scales that occur before the diffusion takes over and turns the kinetic energy over to heat by internal friction.

If we mix 1kg of water with a mixer with an output of 100W, this power would disappear in the smallest vortices, and consequently the diameter of these vortices is of the order

$$l' = \left(\left(\frac{10^{-6}\text{m}^2}{\text{s}}\right)^3 / \left(100\frac{\text{kgm}^2}{\text{s}^2\cdot\text{s}\cdot 1\text{kg}}\right)\right)^{1/4} = 10^{-5}\text{m} = 0.01\text{mm}. \tag{106}$$

2.4.2 Geometric Similarity of Animals

Why do we look like we do? It has long been known that animal forms are not just random, but a result of the strength of muscles and bones in relation to the strength of gravity here on Earth. If we could reduce a human to Thumbelina-size, it turns out that the body would immediately be worn to pieces by the muscles. Therefore, insects usually have very small muscles (thin legs!) in relation to there size.

The discussion below is taken from the note *Dimensional Analysis* of Professor Kristian B. Dysthe, University of Tromsø, 1992. One of his references is the world famous book *On Growth and Form* by D'Arcy W. Thompson, first published in 1917.

We shall first look at animals approximately geometrically similar, and having a typical length scale L. We may then argue that their

1. weight is proportional to their volume, that is, $\propto L^3$
2. muscle power is proportional to the amount of muscle fibers, which in turn is proportional to the muscle cross-sectional area, $\propto L^2$
3. ability to do work (and produce heat), power $\propto$ lung capacity $\propto$ oxygen uptake $\propto$ surface of the lungs $\propto L^2$ (may be somewhat questionable because of the fractal structure of the lung)

Jumping. When an animal wants to jump into the air, it must produce a certain amount of energy which becomes its kinetic energy the moment it leaves the ground. The energy is produced by accelerating the body over a distance, $\mathcal{O}(L)$, multiplied by the power it generates, $\mathcal{O}(L^2)$. In other words, the supply of kinetic energy = force × distance $\propto \mathcal{O}(L^2) \times \mathcal{O}(L) = \mathcal{O}(L^3)$. The necessary potential energy for a jump of height H will likewise be $H = Hmg \propto HL^3$, where m is the mass of the animal. This therefore gives

$$HL^3 = \text{const.} \cdot L^3, \tag{107}$$

or that H is constant. Thus, we get the somewhat surprising result that all animals of the same shape jump equally high!

It should also be similar to jump down a certain height H and land in a controlled manner. Cats seem to have this property, and otherwise it is alleged that the kangaroo and a jumping mouse (which by their names would have slightly different size!) can jump equally high.

Running uphill. From the observation above, the power an animal manages to maintain will be proportional to L^2, and since the required power to keep a speed v up a hill with a slope angle α is

$$(mg\sin\alpha) \cdot v \propto L^3\, v, \tag{108}$$

we obtain

$$v\, L^3 \;=\; \text{const.} \times L^2, \tag{109}$$

or

$$v \;\propto\; 1/L. \tag{110}$$

Small animals can therefore keep a higher speed than large animals when running uphill.

Diving Animals. Assume that all animals during a dive are moving at speed v. The friction force that must be overcome will typically be proportional to the square of the velocity and cross sectional area of the animal, i.e. $F \propto v^2L^2$ (This may be concluded from formulas derived from dimensional analysis). The total energy consumed within the water is $F \cdot (v \cdot t_{\max})$, where $t_{\max}$ denotes the maximum time it can stay under water. Since the energy stored in the animal will be proportional to L^3, then

$$L^2 \cdot t_{\max} \propto L^3, \tag{111}$$

or

$$t_{\max} \propto L. \tag{112}$$

This means, in other words, that large animals can stay longer under water than small animals, and this we know from the marine mammals.

We leave to the reader to speculate about any other scaling arguments, for example, what the Mars and Jupiter residents look like.

Finally, we shall consider two examples from sport.

Weightlifting. For equally shaped weight lifters, the muscle strength is proportional to L^2 and the weight is proportional to L^3. In other words, the force should be proportional to the weight of power 2/3. Figure 9 shows that this holds astonishingly well.

Rowing. We consider boats with the same shape and the typical length L. The necessary input power (due to friction and wave resistance, called *drag*) to maintain a speed $v_{\max}$ is the same here as for diving animals,

$$v_{\max} \cdot F \;\propto\; v_{\max}^3 L^2. \tag{113}$$

The available power is proportional to the number of rowers, N and the displaced fluid volume of the boat is also proportional to N or L^3. Thus

$$L \propto N^{1/3}, \tag{114}$$

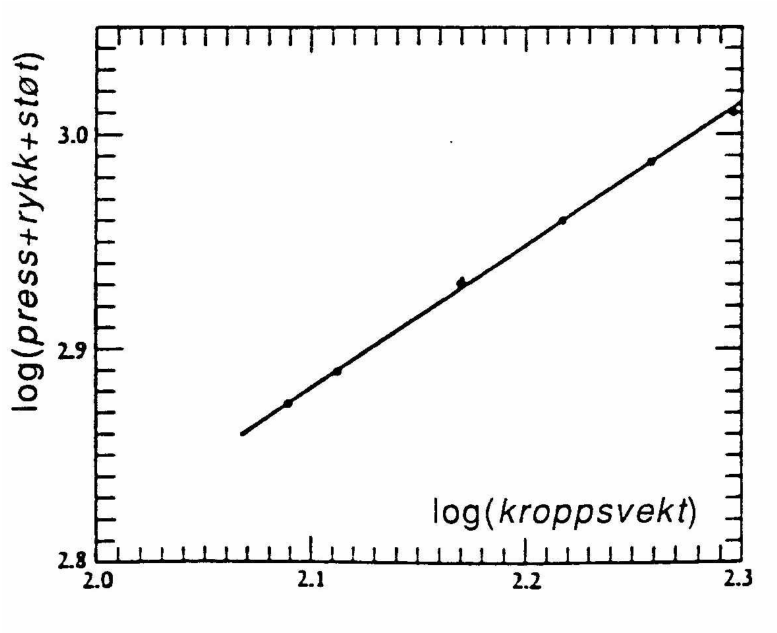

Figure 9: The world records in Snatch + Jerk + Press as a function of the lifter's body weight (kroppsvekt). Note that the line has slope 2/3 (The origin of the figure is unknown).

and we obtain

$$v_{\max}^3 L^2 \propto N, \tag{115}$$

or

$$v_{\max} \propto N^{1/9}. \tag{116}$$

It is surprising that the rate increases so slowly with the number of rowers, but Fig. 10 shows that this indeed seems to be the case.

3 REGULAR PERTURBATION

In this section we shall consider a way to handle equations containing small parameters, and the scaled equations from Case B and Case C in the case study in Sec. 2.3 are of this form. The basic idea is to write the solution as a power series in the small parameter and determine the terms in the series recursively. We shall take a closer look at this methodology and show how it works on some simple examples. Regular perturbation is one of the most common techniques in traditional applied mathematics, and is well treated in several textbooks. The presentation below is incomplete, but adapted to what we are going to need.

We have a *regular perturbation problem* if we have an equation

$$D(x, \cdots, \varepsilon) = 0, \tag{117}$$

containing a small parameter ε so that the full solution, x_{sol}, approaches solution x_0 of the reduced equation

$$D(x, \cdots, 0) = 0, \tag{118}$$

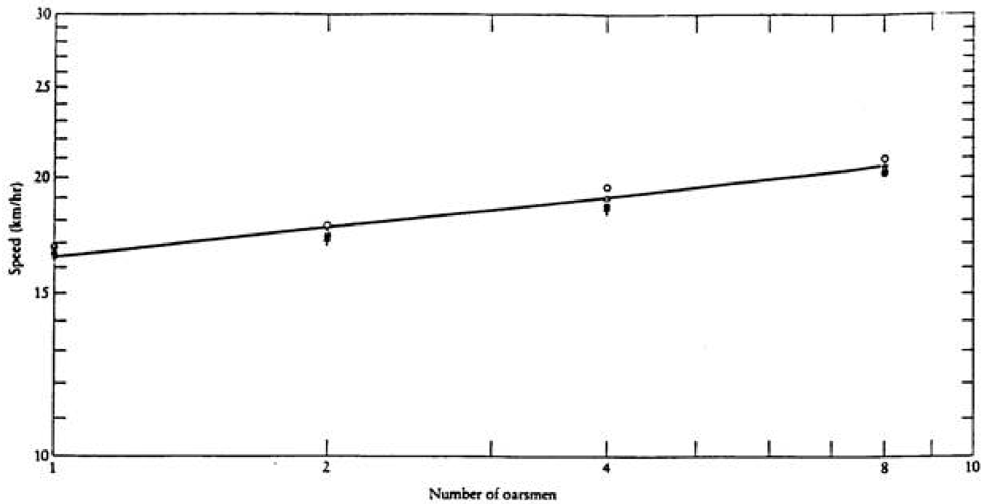

Figure 10: Speed as a function of the number of rowers. The line has slope equal to 1/9. The origin of the graph is unknown.

when ε tends to 0. The statement is somewhat imprecise, as we say nothing about how x_{sol} approaches x_0. If we then know that ε is small (after the equation is scaled), we may approximate the complete solution x_{sol} with x_0. This is pursued further by writing the solution in the form of a power series in ε,

$$x_{sol} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots . \tag{119}$$

for then to come up with a sequence of simpler equations for x_0, x_1, $\cdots$. Since ε is small, we expect that the terms in the series become smaller and smaller, and that the approximation gets better the more terms we include. In practice, it is not that easy. The solution of the equations for x_i often gets more complicated as i increases, and power series do not tend to have very impressive convergence properties.

If we forget about these objections, the method of regular perturbation is easy to state:

1. Write the solution as a power series in ε,

$$x_{sol} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^2 x_3 + \cdots \tag{120}$$

2. Put the series into the equation and clean up the expression so that we obtain a new power series in ε,

$$\begin{aligned} D(x_{sol}, \varepsilon) &= D(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^2 x_3 + \cdots, \varepsilon), \\ &= P(x_0, 0) + P_1(x_0, x_1)\varepsilon + P_2(x_0, x_1, x_2)\varepsilon^2 + \cdots \end{aligned} \tag{121}$$

3. Set each coefficient in the series equal to 0 and solve the equations you then get *recursively*:

$$\begin{aligned} P_0(x_0, 0) &= 0, \\ P_1(x_0, x_1) &= 0, \\ P_2(x_0, x_1, x_2) &= 0, \\ &\cdots \end{aligned} \tag{122}$$

This method gives us $x_0,\ x_1,\ x_2, \cdots$, and the idea may be used in many connections:

- For approximate solutions to algebraic and transcendental equations
- For approximate expressions to integrals
- For ordinary and partial differential equations

Perturbation analysis is often complementary to numerical techniques. In many situations, numerical methods have problems when ε is small (this is especially the case for *singular perturbation* discussed later). The perturbation analysis gives us the asymptotic relations which are useful when ε goes to 0, in contrast to a small number of numerical calculations where we need to keep ε fixed for each calculation. In other contexts there is no really small (or large) parameter to use, and there is no way around numerical calculations.

Perturbation analysis had its best days before we had computers with opportunities for large scale numerical calculations. In particular in the field of aerodynamics and other fluid mechanics. perturbation analysis has been widely recognized. Today, there are computer programs for symbolic manipulation that enables us to find perturbation solutions of orders we could only dream about. However, sometimes profit is marginal, – if one does not achieve reasonable approximations with one or two terms, there is often little to gain by calculating more terms.

3.1 The Projectile Problem

The Projectile Problem, discussed in the book of Lin & Segel [3], pp. 233, is a simple and instructive example of how regular perturbation works. The problem leads to a non-linear differential equation where it is not possible to write the solution in explicit form using elementary functions.

3.1.1 The Model

A projectile is sent vertically up from a planet without atmosphere. The motion is described by the position $x^*(t^*)$ above the planet's surface, where t^* is the time and

$$x^*(0) = 0,\ \frac{dx^*}{dt^*}(0) = V. \tag{123}$$

The projectile will be affected by a force given by Newton's law of gravitation,

$$F(x^*) = -G\frac{Mm}{(R + x^*)^2}, \tag{124}$$

where G is the gravitational constant, M is the planet's mass, R the planet's radius, and m the projectile's mass. Similar to Earth, the gravity force on the planet's surface may be written as $F(0) = -mg$, so that $g = GM/R^2$. Thus, it follows that

$$m\frac{d^2x^*}{dt^{*2}} = -\frac{R^2gm}{(R + x^*)^2}. \tag{125}$$

The mathematical model thus consists of the non-linear differential equation

$$\frac{d^2x^*}{dt^{*2}} = -\frac{R^2g}{(R + x^*)^2}, \tag{126}$$

with the initial conditions stated in Eq. 123.

3.1.2 Scaling

We are going to study a situation where V is much smaller than the planet's *escape velocity*. If V is larger than the escape velocity, the projective will leave the planet permanently. For Earth the escape velocity is about 11.2km/s. However, here the assumption implies that $x^*(t^*) \ll R$ for the whole trip of the projectile. Under this assumption, the equation simplifies to

$$\frac{d^2x^*}{dt^{*2}} = -\frac{R^2 g}{(R+x^*)^2} = -\frac{g}{(1+x^*/R)^2} \approx -g. \tag{127}$$

This equation may easily be solved with the given initial conditions:

$$x^*(t^*) \approx -\frac{1}{2}gt^{*2} + Vt^*. \tag{128}$$

The approximate maximum height follows from Eq. 128 by observing that the time to maximum height is approximately given by

$$\frac{dx^*}{dt^*} \approx -gt^* + V = 0, \tag{129}$$

or

$$t_{\max} \approx \frac{V}{g}. \tag{130}$$

Thus,

$$x_{\max} \approx -\frac{1}{2}g\left(\frac{V}{g}\right)^2 + V\left(\frac{V}{g}\right) = \frac{1}{2}\frac{V^2}{g}. \tag{131}$$

Reasonable scales for Eq. 125, where we do not care about factors of 2, will now be

$$X = \frac{V^2}{g},\ T = \frac{V}{g}. \tag{132}$$

Inserted into the equation, this leads to

$$\frac{d^2\left(\frac{V^2}{g}x\right)}{d\left(\frac{V}{g}t\right)^2} = -\frac{R^2 g}{\left(R+\frac{V^2}{g}x\right)^2}, \tag{133}$$

$$\frac{V^2}{g}x(0) = 0,\ \frac{d\left(\frac{V^2}{g}x\right)}{d\left(\frac{V}{g}t\right)} = V, \tag{134}$$

and after cleaning up, we have the scaled equation:

$$\ddot{x} = -\frac{1}{(1+\varepsilon x)^2}, \tag{135}$$

$$x(0) = 0,\ \dot{x}(0) = 1,\ \varepsilon = \frac{V^2}{Rg}. \tag{136}$$

It turns out that it is not possible to express the solution of this equation, $x = x(t,\varepsilon)$, by means of elementary functions (this is not quite obvious!). Note that since the parameter ε is approximately equal to $2x_{\max}/R$, it is indeed small under the assumption we made above.

3.1.3 Solution by Means of Regular Perturbation

We shall solve the equation 135 using regular perturbation according to the recipe above, and we start by putting

$$x\left(t\right)=x_0\left(t\right)+\varepsilon x_1\left(t\right)+\varepsilon^2 x_2\left(t\right)+\cdots \tag{137}$$

into the equation:

$$\begin{aligned}
\ddot{x}=\ddot{x}_0+\varepsilon\ddot{x}_1+\varepsilon^2\ddot{x}_2+\cdots &= -(1+\varepsilon x)^{-2}\\
&= -\left[1+(-2)\varepsilon x+\frac{(-2)(-3)}{2}(\varepsilon x)^2+\cdots\right]\\
&= -1+2\varepsilon(x_0+\varepsilon x_1+...)-3\varepsilon^2 x_0^2+\cdots\\
&= -1+\varepsilon 2x_0+\varepsilon^2(2x_1-3x_0^2)+\cdots
\end{aligned} \tag{138}$$

(Note the use of Newton's binomial theorem). By collecting the coefficients in front of each power of ε, we find the system

$$\begin{aligned}
\ddot{x}_0 &= -1,\\
\ddot{x}_1 &= 2x_0,\\
\ddot{x}_2 &= 2x_1-3x_0^2,\\
\ddot{x}_3 &= 2x_2+2x_0x_1-2x_0\left(2x_1+x_0^2\right)-2\left(2x_1-3x_0^2\right)x_0,\\
&\cdots
\end{aligned} \tag{139}$$

To find the last equation, we had to expand the series in Eq. 137 to order ε^3. We must also decide what to do with the initial conditions, but here it is reasonable to use

$$\begin{aligned}
x_0\left(0\right)=0,\ \dot{x}_0\left(0\right)=1,\\
x_1\left(0\right)=0,\ \dot{x}_1\left(0\right)=0,\\
x_2\left(0\right)=0,\ \dot{x}_2\left(0\right)=0,\\
\cdots
\end{aligned} \tag{140}$$

Thus, x_0 takes care of the initial conditions, which are consequently satisfied no matter where we stop the series expansion. The solution for x_0 follows immediately from equation 139:

$$x_0\left(t\right)=t-\frac{1}{2}t^2, \tag{141}$$

and by introducing this into the next equation, in 139, we find

$$\ddot{x}_1=2x_0=2\left(t-\frac{1}{2}t^2\right), \tag{142}$$

or

$$x_1\left(t\right)=\frac{1}{3}t^3-\frac{1}{12}t^4. \tag{143}$$

Note that only the particular solutions change with every step, and that the contribution from the homogeneous solutions disappear for $x_i\left(t\right)$ when $i\geq 1$.

Commonly, the algebra quickly becomes quite complicated, but today we can make good use of software for symbolic manipulation, such as *Maple*, *MuPad*, *Mathematica,* or the free *wxMaxima* (*Maple* was used here):

$$x(t) = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right) + \mathcal{O}\left(\varepsilon^3\right). \tag{144}$$

From this solution, we also find a more accurate equation for the time to the maximum,

$$\begin{aligned} &\frac{d}{dt}\left(t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right)\right) \\ &= 1 - t + \varepsilon t^2 - \frac{1}{3}\varepsilon t^3 - \varepsilon^2 t^3 + \frac{11}{12}\varepsilon^2 t^4 - \frac{11}{60}\varepsilon^2 t^5 = 0. \end{aligned} \tag{145}$$

This is a fifth-degree equation, but since we expect the solution t_m to be close to 1, we can, in accordance with the foregoing, try a perturbation expansion:

$$t_m = 1 + a\varepsilon + b\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{146}$$

By introducing this into Eq. 145, we find

$$\begin{aligned} 0 &= 1 - \left(1 + a\varepsilon + b\varepsilon^2\right) + \varepsilon\left(1 + a\varepsilon + b\varepsilon^2\right)^2 - \frac{1}{3}\varepsilon\left(1 + a\varepsilon + b\varepsilon^2\right)^3 - \varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^3 + \\ &+ \frac{11}{12}\varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^4 - \frac{11}{60}\varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^5 + \cdots \\ &= \left(-a + \frac{2}{3}\right)\varepsilon + \left(a - \frac{4}{15} - b\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right). \end{aligned} \tag{147}$$

This gives to $\mathcal{O}\left(\varepsilon^2\right)$

$$a = \frac{2}{3},\ b = a - \frac{4}{15} = \frac{2}{5}, \tag{148}$$

and

$$t_m = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{149}$$

It is reasonable that the time it takes up to a maximum increases a little from 1, since gravity acting on the projectile becomes weaker as it rises. The figure 11 shows some numerical solutions created using *Matlab*^{TM}.

The perturbation expansion to zeroth, first and second order are compared to the numerical solution on the figures 12 and 13.

3.1.4 Analytical Solution

As remarked above, is not possible to express the full solution of

$$\ddot{x} = -\frac{1}{(1 + \varepsilon x)^2},\ \ x(0) = 0,\ \ \dot{x}(0) = 1, \tag{150}$$

in closed form by means of elementary functions. However, it is possible to do something. After multiplying the equation with $\dot{x}$,we find

$$\frac{d}{dt}\left(\dot{x}^2/2\right) = \frac{d}{dt}\left(\frac{1}{\varepsilon}\frac{1}{1 + \varepsilon x}\right), \tag{151}$$

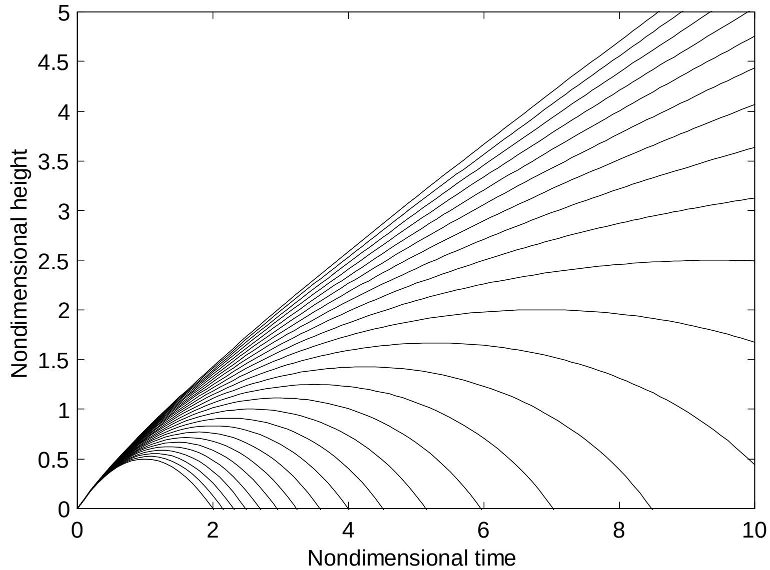

Figure 11: Numerical solutions shown for $\varepsilon = 0\,(0.1)\,3$. When $\varepsilon \geq 2$, the initial speed is above the escape speed, and the projectile never returns to $x = 0$.

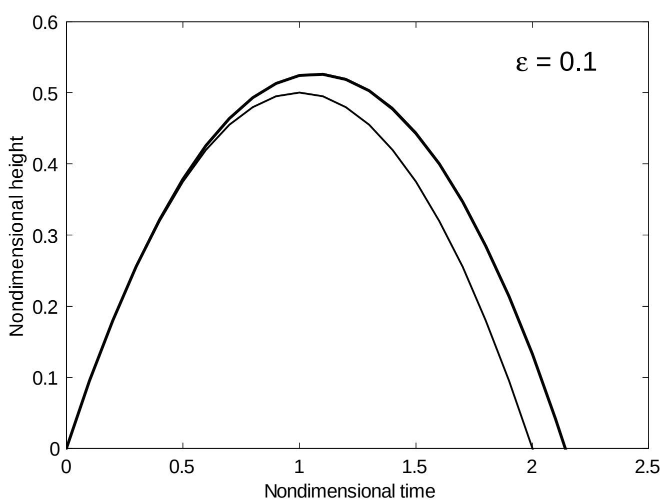

Figure 12: Numerical solution and perturbation solutions for $\varepsilon = 0.1$. The numerical solution (thick line) and the perturbation solutions to first and second order collapse on the graph. The solution for x_1 is different, however.

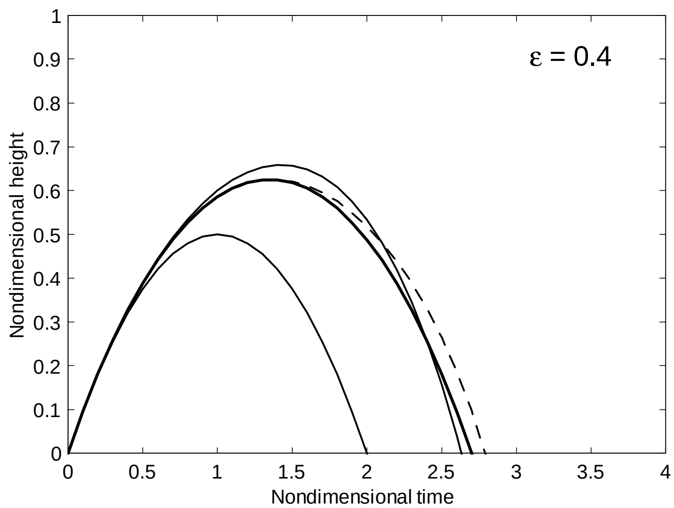

Figure 13: Similar to previous figure for $\varepsilon = 0.4$. Numerical solution: thick curve; x_0 and $x_0 + \varepsilon x_1$: thin curve; $x_0 + \varepsilon x_1 + \varepsilon^2 x_2$: dashed curve.

or that

$$\frac{\dot{x}^2}{2} - \frac{1}{\varepsilon}\frac{1}{1+\varepsilon x} = \text{ constant.} \tag{152}$$

This tells us that the motion is conservative, such that the sum of the potential and kinetic energies is constant. By introducing the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$, we find that the constant is equal to $1/2 - 1/\varepsilon$. This leads to a first order non-linear equation:

$$\dot{x}^2 = \frac{1 + (\varepsilon - 2)x}{1 + \varepsilon x}, \quad x(0) = 0. \tag{153}$$

If $\varepsilon < 2$, $\dot{x}$ will be 0 for

$$x_{\text{max}} = \frac{1}{2-\varepsilon}. \tag{154}$$

This is therefore the exact expression for the maximum height of the projectile when $\varepsilon < 2$. If $\varepsilon > 2$, the speed will always be greater than 0 and the projectile continues to the boundaries of the universe. Note that for Earth, $\varepsilon = 2$ means that

$$V = \sqrt{2Rg} = \sqrt{2 \times \frac{40000000}{2\pi} \times 9.81}\frac{\text{m}}{\text{s}} \approx 11.17\text{km/s}, \tag{155}$$

which is therefore the escape velocity referred to above.

By taking the square root of Eq. 153 and separating the variables, and as long as $\dot{x} \geq 0$, we may write the solution of Eq. 153, implicitly as

$$t = \int_0^x \sqrt{\frac{1+\varepsilon s}{1+(\varepsilon-2)s}} ds. \tag{156}$$

This integral turns out to be solvable,

$$\int \sqrt{\frac{a+s}{b-s}} ds = \frac{a+b}{2} \arcsin\left(\frac{2s+a-b}{a+b}\right) - \sqrt{(a+s)(b-s)} + C = F(s,a,b) + C. \tag{157}$$

Thus,

$$t = \int_0^x \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}} ds = \sqrt{\frac{\varepsilon}{2-\varepsilon}} \int_0^x \sqrt{\frac{1/\varepsilon + \varepsilon s}{1/(2-\varepsilon) - s}} ds$$
$$= \sqrt{\frac{\varepsilon}{2-\varepsilon}} \left[F(x, \frac{1}{\varepsilon}, \frac{1}{2-\varepsilon}) - F(0, \frac{1}{\varepsilon}, \frac{1}{2-\varepsilon}) \right]. \tag{158}$$

Since we already know that

$$x_{\max} = \frac{1}{2-\varepsilon}, \tag{159}$$

when $\varepsilon < 2$, we also find an exact expression for t_m:

$$t_m = \int_{s=0}^{1/(2-\varepsilon)} \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}} ds = \frac{\frac{\pi}{2} - \arcsin(1-\varepsilon) + \sqrt{(2-\varepsilon)\varepsilon}}{(2-\varepsilon)^{3/2}\varepsilon^{1/2}}$$
$$= 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + \frac{8}{35}\varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right). \tag{160}$$

The start of the power series is similar to what we found in Eq. 149.

3.2 Florence Griffith Joyner and the World Record in 100 meters

Florence Griffith Joyner, "Flo-Jo" (1959–98) was an American track runner who is still (as of this writing) the holder of the official world record in 100 meters, 10.49s. The record was set during a quarter-final of the US qualifying heats for the Seoul Olympics in 1988. The wind gauge registered 0 m/s, while many argued that there was considerable tailwind, estimated to about 4 m/s, and that the meter did not work. In the rest of the qualifying races she ran on times around 10.7s.

A sprinter is dependent on the pushing force she/he is able to produce. This force may be written Mp^*, where M is the runner's mass and p^* a parameter with the unit of acceleration. The maximum pushing force is thus MP, where P is the maximum p^* the runner is able to produce.

In addition, there are two forces slowing the sprinter: *air resistance* and *internal friction*. The internal friction, which represents the resistance of muscles and joints, is believed to be written in the form Mu^*/τ, where u^* is the runner's speed and τ is a characteristic time constant. Measurements of different runners, including Ben Johnson and Carl Lewis, have given $P \approx 10\text{m/s}^2$ and $\tau \approx 1$s, and we shall use these values below.

Based on Newton's second law, we can now write the equation of motion for the sprinter

$$M\frac{du^*}{dt^*} = Mp^*(t^*) - M\frac{u^*}{\tau} - F_l, \tag{161}$$

where F_1 represents the air resistance. The expression for the air resistance may be found by dimensional analysis. It is reasonable to assume that F_l depends on the air density, ρ_{air}, airs kinematic viscosity ν, the runner's velocity u^*, and the runner's cross-sectional area A in the direction of motion. In addition, we need a length scale L, for which we may use $\sqrt{A}$. It is left to the reader to show that from dimensional analysis we may now write

$$F_l = \frac{1}{2}\rho_{luft} C_D \left(Re\right) A u^{*2}, \tag{162}$$

where C_D is an unknown function, the so-called *drag coefficient*, which depends on *Reynolds number*, *Re*= $\frac{\sqrt{A}u^*}{\nu}$.

Let us now scale Eq. 161. Since we already know that τ is a typical time constant, we decide to use this as our time scale. We know the maximum P, and because of its unit, this is a natural scale for the acceleration. Thus, we scale the velocity by using $P\tau$ and obtain

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t), \tag{163}$$

where

$$\varepsilon = \tfrac{1}{2}\rho_l C_D \left(\frac{\sqrt{A}u^*}{\nu}\right) \tau^2 P \tfrac{A}{M}. \tag{164}$$

Now C_D itself is depending on the speed, but measurements of air resistance for irregular bodies have shown that C_D is almost constant for $2 \times 10^4 < Re < 10^6$, which covers mainly what we are facing here. This value of C_D is close to 1, which we for simplicity shall use below. With P and τ given as above, $A \approx 0.45\text{m}^2$ and $\rho_{air} \approx 1.2\text{kg/m}$, ε is thus about 0.035 for an athlete weighing 70 to 80 kg. Since $|u| \leq 1$, we conclude that the air resistance is a relatively small term in the equation.

In order to solve 163 by regular perturbation, we write

$$u(t) = u_0(t) + \varepsilon u_1(t) + \mathcal{O}(\varepsilon^2) \tag{165}$$

and put this in 163:

$$\dot{u}_0 + \varepsilon\dot{u}_1 + \mathcal{O}(\varepsilon^2) + u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2) + \varepsilon(u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2))^2 = p. \tag{166}$$

We collect all parts of the same order in ε and use $u(0) = 0$ as the initial condition. This gives us a sequence of first order equations:

$$\begin{aligned} &\dot{u}_0 + u_0 = p,\ u_0(0) = 0, \\ &\dot{u}_1 + u_1 = -u_0^2,\ u_1(0) = 0, \\ &\dot{u}_2 + u_2 = -2u_0u_1,\ u_2(0) = 0, \\ &\qquad\vdots \end{aligned} \tag{167}$$

In order to solve the equations, we must also decide what to use for the acceleration $p(t)$. Let us for simplicity assume that $p^*(t) = P$, that is, $p(t) \equiv 1$. The first two terms of perturbation expansion are thus determined by

$$\begin{aligned} &\dot{u}_0 + u_0 = 1,\ u_0(0) = 0, \\ &\dot{u}_1 + u_1 = -u_0^2,\ u_1(0) = 0, \end{aligned} \tag{168}$$

and we easily find that the solution to order ε is

$$u(t) = 1 - e^{-t} + \varepsilon[-1 + 2te^{-t} + e^{-2t}] + \mathcal{O}\left(\varepsilon^2\right). \tag{169}$$

By plotting the graph we see that the sprinter reaches maximum velocity ($u_0(t) \approx 1$) approximately at $t = 3$, that is, after $3\tau = 3$ seconds.

Let us now assume that we have a winds blowing parallel to the running direction. This leads to a modified drag

$$F_l = \frac{1}{2}\rho_l C_D A(u^* - W)^2. \tag{170}$$

Wind speed is scaled similar to the runner's speed, so that in dimensionless variables the dimensionless wind is given by $\delta = W/(\tau P)$.

Let us determine the maximum velocity U a sprinter can hold as a function of δ to the first order in ε. The maximum speed is achieved when the acceleration is zero, i.e., given by the equation

$$U + \varepsilon(U - \delta)^2 = 1. \tag{171}$$

Check that the solution to first order in ε is

$$U = 1 - \varepsilon(1 - \delta)^2 + \mathcal{O}(\varepsilon^2). \tag{172}$$

We assume that Florence performed her maximum in all races and (somewhat unrealistically) that she had the maximum speed throughout the race. The maximum speed without wind is $U_0 = 1 - \varepsilon$, while $U_4 = 1 - 0.36\varepsilon$ with a tailwind equal to 4m/s, that is, $\delta = 0.4$. The total time used for 100m will be $T = (100\text{m})/U$, and thus

$$T_0/T_4 = (1 - 0.36\varepsilon)/(1 - \varepsilon). \tag{173}$$

To get an idea of what this means in time, we must find "her" ε. We assume that she has the same acceleration, cross-sectional area and drag coefficient as the one above. Her weight, however, should be somewhat less, let's say 60 kg. That gives $\varepsilon \approx 0.04$. Expressed in time 4 m/s tailwind gives about

$$10.7\text{s} \cdot \frac{1 - 0.04}{1 - 0.04 \times 0.36} = 10.42\text{s}. \tag{174}$$

As she accelerated the first three seconds, this fits very well with the time she actually used. It should also be noted that she ran on 10.54s in the finals, but then the tailwind was about 2m/s (Was that reasonable?).

3.3 Modeling the Kidney Function

This modeling example is also thoroughly covered in [3], Chapter 8, The example deals with the whole process of physical understanding, formulation of mathematical model, scaling, and finally regular perturbation. In the modeling we meet conservation principles that will also be important later in the course.

After observations of kidney tissue, J. Diamond suggested in 1967 that salt and water are expelled from the kidneys in an indirect way by means of *secondary channels*. Such channels have been found in the walls of the main channels, and it is speculated that this might explain how kidneys work. The proposed mechanism is outlined in Figure 14. At the inner end of the secondary channels there are chemical pumps that send salt into the channel with the consumption of chemical energy. Consequently, the solution in the channel has higher salt concentration than the surrounding tissue and the main channel. This means that there is a gradient in the salt concentration towards the opening. Concentration difference between the channel and the

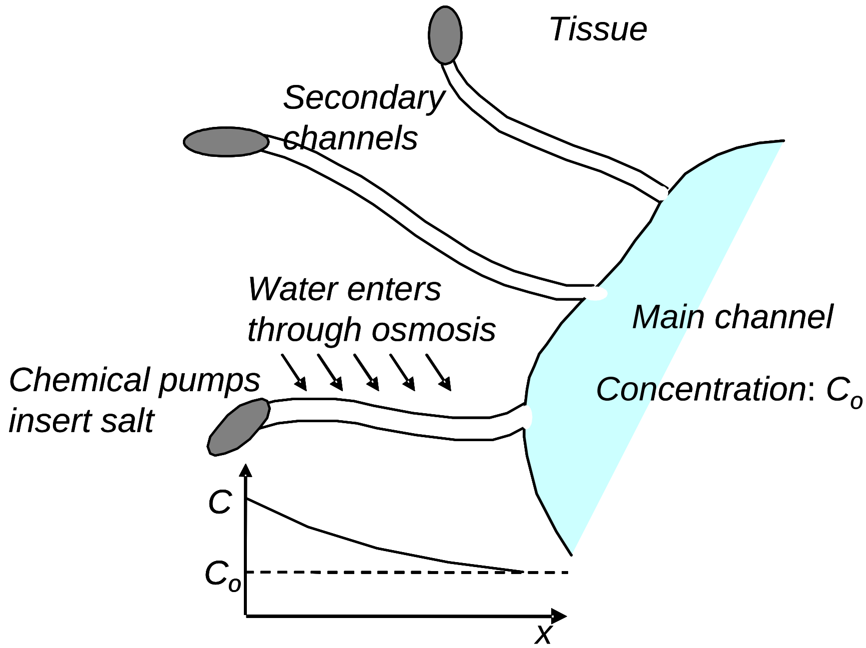

Figure 14: Sketch showing how one thinks that the kidneys are functioning.

surrounding tissue causes the water to enter the channel through the walls because of *osmosis*. The water flowing into the main channel transports at the same time the salt out of the channel. Under stationary conditions we get a so-called *standing gradient* in the salt solution in the channel. Actually, salt is transported in the channel both by *diffusion* and by *convective* transport by the water flow. The problem we are facing, is to set up a mathematical model for a secondary channel and determine how the salt concentration and water flow vary in the channel, and, in particular, how effective the proposed mechanism might be.

The description above includes a number of concepts that we first need to consider. Molecular *osmosis* is an important mechanism in biological systems. Osmosis involves transport through membranes that are so finely meshed that they do not allow large molecules to pass through the channel. In our case the channel wall is modeled as an osmotic membrane, sketched in Fig. 15.

If the ion concentrations on the sides of the membrane are C_1 and C_2 respectively, the net amount of water passing through the membrane per area and unit time, J, may be written as

$$J = P(C_2 - C_1). \tag{175}$$

The constant P is called *permeability*. The units of the variables in this equation are as follows:

$$\begin{aligned} [J] &= \frac{\text{Volume}}{\text{Area} \times \text{Time}} = \frac{\text{m}^3}{\text{m}^2\text{s}} = \frac{\text{m}}{\text{s}}, \\ [C] &= \frac{\#\text{ ions}}{\text{Volume}} = \frac{\text{osmol}}{\text{m}^3}, \\ [P] &= \frac{\text{m/s}}{\text{osmol/m}^3} = \frac{\text{m}^4}{\text{osmol} \cdot \text{s}}. \end{aligned} \tag{176}$$

The unit *osmol* is used in [3], but is actually not necessary since the number of ions for thin

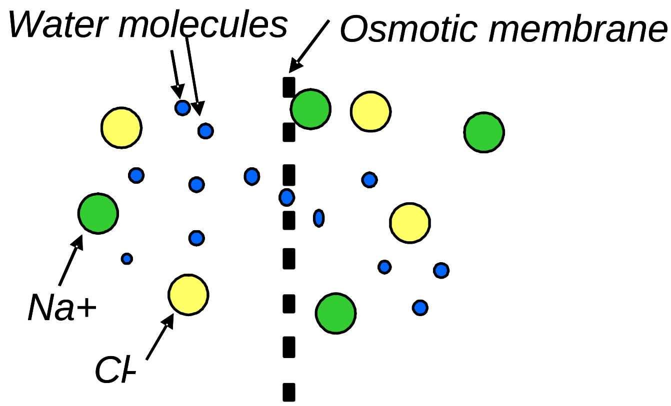

Figure 15: An osmotic membrane in a salt solution lets water molecules through, but not Chlorine and Sodium ions.

salt solutions will be twice the number of salt molecules, and therefore proportional to the salt concentration measured in kg/m^3.

We shall later return to the concepts of *diffusion* and *flux.* The transport of dissolved salt in a solution that is otherwise at rest, is mainly due to concentration differences:

$$F = -D\frac{\partial C}{\partial x}. \tag{177}$$

Here F is called *diffusive flux of* salt (in the x-direction), and D is called the diffusion coefficient.

Flux is in general the amount that passes through an imaginary surface per time and unit area. The unit of flux of salt is thus

$$[F] = \frac{\text{Amount}}{\text{Area} \times \text{Time}} = \frac{\text{osmol}}{\text{m}^2\text{s}}, \tag{178}$$

and since $[C]$ =osmol/m^3, then, from Eq.177,

$$[D] = \frac{\text{m}^2}{\text{s}}. \tag{179}$$

If a salt solution with concentration C is moving in the x-direction with velocity V, the expression for the *convective flux* is derived by considering Fig. 16. During one second the shaded volume passes through the surface with area A.

The expression for the flux is thus

$$F_{konv} = \frac{V \cdot 1s \cdot A \cdot C}{A \cdot 1s} = VC. \tag{180}$$

The total flux will be the sum of the contributions (this will be discussed in more depth later):

$$F = F_{diff} + F_{konv} = -D\frac{dC}{dx} + VC. \tag{181}$$

Chemical pumps are able to push the saline through the tissue (the body) by the consumption of chemical energy (that is about all that this author knows about it). The strength of a chemical pump is defined by its strength, N_0,

$$[N_0] = \frac{\text{Amount}}{\text{Wall area} \times \text{Time}} = \frac{\text{osmol}}{\text{m}^2\text{s}}. \tag{182}$$

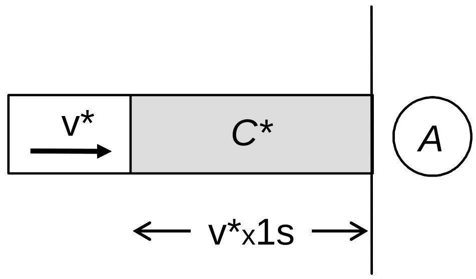

Figure 16: *Convective flux*: The solution with a salt concentration C^* is moving with speed v^* in the x-direction.

3.3.1 Formulation of the mathematical model

Figure 17 is a sketch of how we can imagine a one-dimensional mathematical model of the channel in the form of a straight tube of length, L, cross section, A, radius, c, and an active zone for the chemical pumps of length δ. We need to ask the biomedical and physical expertise to get an idea forthe size of the variables we have introduced. We shall return to the scaling below, but it is already here worth noting that the channels are *thin* (the diameter is only 0,001–0.01 times their length), so it must be sufficient to imagine a *one-dimensional* model.

As usual, we let non-scaled variables have an extra *, which is removed after the scaling. The water coming in through the wall (per unit area) due to osmosis at the location x^* will be

$$J = P\left[C^*(x^*) - C_0\right]. \tag{183}$$

The salt in the channel is transported both by diffusion and convection, and therefore

$$F^* = F_d + F_c = v^*C^* - D\frac{dC^*}{dx^*}. \tag{184}$$

The chemical pumps enter an amount of salt per time unit equal to $N_0(\delta c)$.

We have been asked to determine the so-called *emergent osmolarity*, Os^*, defined by $F^*(L) = v^*(L)Os^*$:

$$Os^* = \frac{F^*(L)}{v^*(L)} = \frac{v^*(L)C^*(L) - D\frac{dC^*}{dx^*}(L)}{v^*(L)}. \tag{185}$$

This is the equivalent salt concentration that would have the same outflow of salt, if the solution had speed v^* and there was no contribution from diffusion.

The basic principle of the modeling is the *preservation* or *conservation* of salt and water. This is the mathematical concept for the everyday expression that "what goes in must come out". Conservation of water for a part of the channel between x^* and $x^* + \Delta x^*$ can be illustrated as in 18. The terms of Q_{inn} and Q_{ut} (flow of water in and out per time unit) are simple:

$$\begin{aligned} Q_{inn} &= Av^*(x^*), \\ Q_{out} &= Av^*(x^* + \Delta x^*) \end{aligned} \tag{186}$$

(Check this out by a similar argument to the one in Fig. 16). For Q_{os} it must be possible to write

$$Q_{os} = P\left(C^*(x^* + \tilde{x}^*) - C_0\right) \cdot (\Delta x^* \cdot c), \tag{187}$$

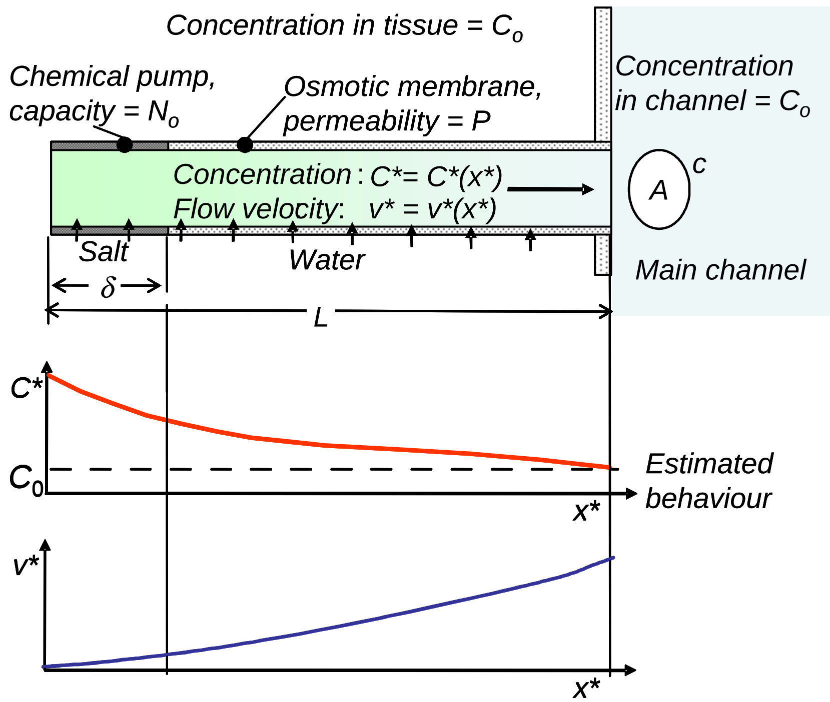

Figure 17: Sketch of one-dimensional mathematical model of the secondary channel.

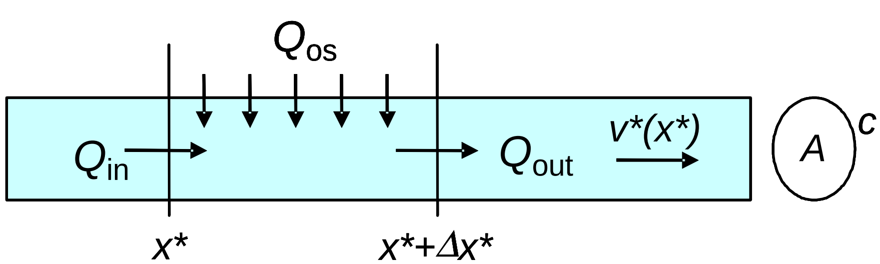

Figure 18: Illustration of the conservation of water for the section of the channel between x^* and $x^*+\Delta x^*$.

where $x^* \leq \tilde{x}^* \leq x^* + \Delta x^*$, and $\Delta x^* \cdot c$ is the wall area. Since $Q_{out} - Q_{in} = Q_{os}$, we have

$$Av^*(x^* + \Delta x^*) - Av^*(x^*) = P\left(C^*(x^* + \tilde{x}^*) - C_0\right) \cdot \left(\Delta x^* \cdot c\right). \tag{188}$$

By dividing by Δx^* and letting $\Delta x^* \to 0$, we find the differential equation

$$\frac{dv^*}{dx^*} = \frac{Pc}{A}\left[C^*(x^*) - C_0\right]. \tag{189}$$

Conservation of the salt can be set up just as easily:

$$\begin{aligned} Q_{ut}^{salt} &= AF^*(x^* + \Delta x^*), \\ Q_{inn}^{salt} &= AF^*(x^*), \\ Q_{k.p.}^{salt} &= N^*(x^*) \cdot (c\Delta x^*), \end{aligned} \tag{190}$$

where $N^*(x^*) = N_0$ when $x^* \leq \delta$, and equal to 0 otherwise. By letting $\Delta x^* \to 0$, we derive in a similar way

$$A\frac{dF^*}{dx^*} = \begin{cases} N_0 c, & x^* < \delta, \\ 0 & \delta \leq x^*. \end{cases} \tag{191}$$

This simple equation can be solved immediately. For $x^* < \delta$ we find, since $F^*(0) = 0$ (we assume that nothing enters through the end surface),

$$F^* = \frac{N_0 c}{A} x^*. \tag{192}$$

For the rest of the channel flux is constant. Since the flux has to be continuous at $x^* = \delta$ (think of what that means!),

$$F^* = \frac{N_0 c}{A}\delta. \tag{193}$$

This gives us now the following differential equation for C^*:

$$C^* v^* - D\frac{dC^*}{dx^*} = \begin{cases} \frac{N_0 c}{A} x^*, & x^* < \delta, \\ \frac{N_0 c}{A}\delta, & \delta \leq x^*. \end{cases} \tag{194}$$

Together with equation 189 we have got two non-linear, coupled differential equations for C^* and v^*. Before we try to solve them, we need to specify boundary conditions for the ends of the channel, as well as the continuity conditions (also called "matching" conditions) at $x^* = \delta$.

At $x^* = 0$ it is clear that

$$v^*(0) = 0, \; F^*(0) = 0. \tag{195}$$

(We consider the end of the channel to be closed). This implies, by means of equation 194, that

$$dC^*/dx^*(0) = 0.$$

At the outer end of the channel, it is reasonable to use

$$C^*(L) = C_0. \tag{196}$$

Finally, we consider $x^* = \delta$. It is clear that both F^*, v^* and C^* must be continuous (Think about it. Diffusion ensures that C^* does not develop discontinuities). Thus we may write

$$\begin{aligned} F^*(\delta+) &= F^*(\delta-), \\ v^*(\delta+) &= v^*(\delta-), \\ C^*(\delta+) &= C^*(\delta-). \end{aligned} \tag{197}$$

Quantity	Unit	Min. value	Typical value	Max. value
r	cm	10^{-6}	5×10^{-6}	10^{-4}
L	cm	4×10^{-4}	10^{-2}	2×10^{-2}
δ	cm	4×10^{-5}	10^{-3}	2×10^{-3}
D	cm^2/s	10^{-6}	10^{-5}	5×10^{-5}
N_0	mOsm/cm^{2}s	10^{-10}	10^{-7}	10^{-5}
P	cm^4/s mOsm	10^{-6}	2×10^{-5}	2×10^{-4}
C_0	mOsm/cm^3	–	3×10^{-1}	–

Table 6: Overview of variables and parameters of the model. Here r is the radius of the channel, assumed to have a circular cross section.

Check from Eq. 194 that then also

$$\frac{dC^*}{dx^*}(\delta+) = \frac{dC^*}{dx^*}(\delta-). \tag{198}$$

We sum up the mathematical model:

Equations:

$$\frac{dv^*}{dx^*} = \frac{Pc}{A}\left(C^*(x^*) - C_0\right), \quad 0 \le x^* \le L, \tag{199}$$

$$C^*v^* - D\frac{dC^*}{dx^*} = \begin{cases} \frac{N_0 c}{A}x^*, & x^* < \delta, \\ \frac{N_0 c}{A}\delta & \delta \le x^*. \end{cases} \tag{200}$$

Boundary conditions:

$$v^*(0) = 0, \quad C^*(L) = 0, \tag{201}$$

$$v^*, C^* \text{ is continuous for } x^* = \delta. \tag{202}$$

Determine

$$Os^* = \frac{F^*(L)}{v^*(L)} = \frac{cN_0\delta/A}{v^*(L)}. \tag{203}$$

3.3.2 Scaling

Here we need first to obtain information about the values of variables parameters involved, and Table 6 has been copied from [3] (Table 8.2, p. 264).

We may use both L and δ as the length scale, but choose we follow [3] and use δ. The concentration scale is self-evident, namely C_0, while the velocity scale requires some creativity. Lin and Segel [3] propose to compute the velocity scale U by balancing the amount of salt produced per time unit with the amount leaving the channel by convection if the concentration is C_0. In other words,

$$cN_0\delta = C_0 \times (AU), \tag{204}$$

or

$$U = \frac{cN_0\delta}{C_0 A}. \tag{205}$$

We may then set

$$x^* = \delta x, \quad C^* = C_0 C, \quad v^* = Uv, \tag{206}$$

Dimensionless parameter	Min. value	Typical value	Max. value
ε	10^{-5}	$2{\times}10^{-2}$	10^{2}
η	$4{\times}10^{-3}$	75	10^{10}
λ	10	10	500

Table 7: The size range of the dimensionless parameters.

and leave to the reader to check that we can write the result as

$$\varepsilon\frac{dv}{dx} = C - 1,\ \ 0 \le x \le \lambda,$$
$$Cv - \eta\frac{dC}{dx} = \begin{cases} x, & 0 \le x \le 1 \\ 1 & 1 \le x \le \lambda \end{cases}, \tag{207}$$

with conditions

$$v(0) = 0,\ \ C(\lambda) = 1, \tag{208}$$
$$v,\ C,\ dC/dx \text{ continuous at } x = 1, \tag{209}$$

and the three dimensionless parameters:

$$\varepsilon = \frac{N_0}{PC_0^2},$$
$$\eta = \frac{AC_0D}{N_0\delta^2 c}, \tag{210}$$
$$\lambda = \frac{L}{\delta}.$$

The dimensionless emergent osmolarity then becomes

$$Os = \frac{Os^*}{C_0} = \frac{1}{v(\lambda)}. \tag{211}$$

The range of magnitude of the dimensionless variables can now be estimated from Table 6, and the result is shown in Table 7.

3.3.3 Perturbation Analysis

Our equations do not look difficult at first glance, but they are nonlinear and coupled, and the initial conditions of C and v are given at the opposite ends of the channel.

Since ε will typically be small, it is natural to attempt a perturbation series, and Lin and Segel describes here how they first got stuck by inserting

$$C = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + ...,$$
$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + ... \tag{212}$$

in equations 207 (Note that C_0 is here 0-th order solution for C and not the concentration scale). Check that we obtain $C_0 = 1$ to order ε , and

$$1v_0 + \eta\frac{dC_0}{dx} = \begin{cases} x \\ 1 \end{cases}, \tag{213}$$

that is,

$$v_0 = \begin{cases} x & 0 \le x \le 1, \\ 1 & 1 \le x \le \lambda. \end{cases} \tag{214}$$

This looks good so far, but to order ε we derive

$$C_1(x) = \frac{dv_0}{dx} = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & 1 \le x \le \lambda, \end{cases} \tag{215}$$

in other words, C_1 is discontinuous. Also the perturbation solutions should be continuous!

It is therefore something fundamentally wrong with the assumptions we have made. If we look at the expressions for ε and η, N_0 occurs in the numerator in ε and in the denominator for η. In addition, it appears that N_0 can vary over more than 12 powers of 10. This means that if ε is small, then η tends to be large. Lin and Segel now try to introduce more stable parameter κ defined by

$$\eta = \frac{\left(\lambda^2/\kappa^2\right)}{\varepsilon}. \tag{216}$$

The essential point here is that $\eta \propto \varepsilon^{-1}$, whereas the chosen form simplifies the expressions. If we insert this, we get a set of modified equations,

$$C - 1 = \varepsilon \frac{dv}{dx}, \quad 0 \le x \le \lambda, \tag{217}$$

$$\varepsilon\kappa^2 Cv - \lambda^2 \frac{dC}{dx} = \varepsilon\kappa^2 \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 \le x \le \lambda \end{cases}, \tag{218}$$

$$v(0) = 0, \quad C(\lambda) = 1, \tag{219}$$

$$v,\ C,\ dC/dx \text{ continuous at } x = 1. \tag{220}$$

Moreover,

$$\varepsilon = \frac{N_0}{PC_0^2}, \quad \kappa = \left(\frac{\lambda^2}{\eta\nu}\right)^{1/2} = \left(\frac{cPC_0L^2}{AD}\right)^{1/2}, \ \lambda = \frac{L}{\delta}. \tag{221}$$

Before we start a new perturbation analysis we can test the problem with some numerical experiments. The equations are a first-order system, but to solve them in the standard way, we must start $C(x)$ and $v(x)$ in the same point. If we choose to start at 0, then we need to vary $C(0)$ so that we really hit $C(\lambda) = 1$. This is called solving by a "shooting method". The search for the starting value can be build into the program, but here it is just as easy to use "trial and error". Matlab^{TM}-code needed to solve the equations is simple:

```
x0 =[0, 1.0355]; tspan = [0, 10];
[t,x]=ode45('nyre',tspan,x0);
subplot(2,1,1);plot(t,x(:,2));
subplot(2,1,2);plot(t,x(:,1));
```

The function that defines the equations:

```
function xdot=nyre(t,x)
lambda = 10.;
kappa = 1;
eps = .5;
```

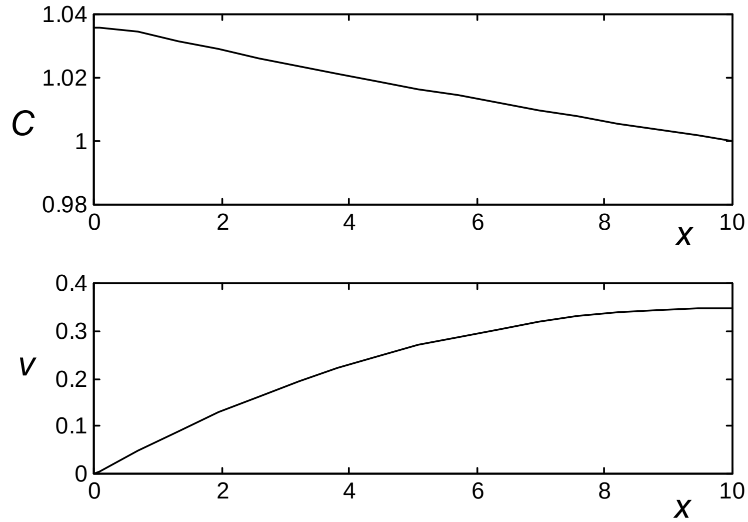

Figure 19: Solution obtained with the "shooting method". Parameter values: $\lambda = 10, \quad \kappa = 1, \quad \varepsilon = .5$. In order to end in 1 for $x = 10$, $C(0)$ must be about 1.0355.

```
A= eps*kappa^2/lambda^2;
xdot(1)=( x(2)-1 )/eps;
xdot(2)=A*x(1)*x(2)-A*min(t,1);
```

A sample result is shown in Figure 19. Let us now look at the modified equations, and try a perturbation analysis as given in Eq. 212. To order ε^0 we obtain as before $C_0 = 1$, but now the equations to order ε are more interesting:

$$C_1 = \frac{dv_0}{dx}, \tag{222}$$

$$\kappa^2 C_0 v_0 - \lambda^2 \frac{dC_1}{dx} = \kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{223}$$

that is,

$$\kappa^2 v_0 - \lambda^2 v_0'' = \kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix} \tag{224}$$

for v_0. This equation must be solved for both the right side and put together so that the boundary condition at 0 and continuity conditions in $x = 1$ holds. We leave it to readers to check the following solution for $0 \leq x \leq 1$:

$$v_0 = x - K_1 \sinh\left(\frac{\kappa}{\lambda}x\right), \tag{225}$$

$$C_1 = v_0' = 1 - K_1 \frac{\kappa}{\lambda} \cosh\left(\frac{\kappa}{\lambda}x\right). \tag{226}$$

The solution for $1 \leq x \leq \lambda$:

$$v_0 = 1 - K_2 \cosh(\frac{\kappa}{\lambda}x - \kappa), \tag{227}$$

$$C_1 = v_0' = -K_2 \frac{\kappa}{\lambda} \sinh\left(\frac{\kappa}{\lambda}x - \kappa\right) \tag{228}$$

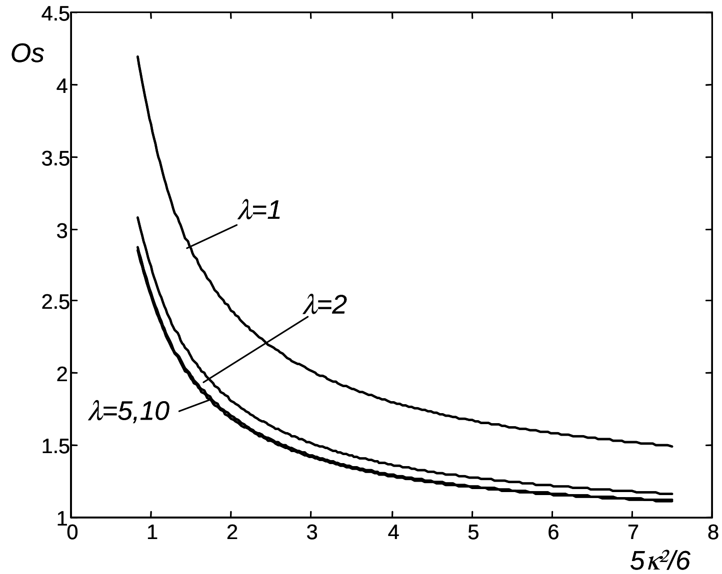

Figure 20: Dimensionless osmolarity to first order in Eq. \ref{os}.

(note that $C_1(\lambda) = 0!$).

The constants K_1 and K_2 are determined by the continuity conditions at $x = 1$:

$$K_1 = \frac{\lambda}{\kappa} \frac{\cosh(\kappa/\lambda - \kappa)}{\cosh(\kappa)}, \tag{229}$$

$$K_2 = \frac{\lambda}{\kappa} \frac{\sinh(\kappa/\lambda)}{\cosh(\kappa)}. \tag{230}$$

The dimensionless osmolarity is approximately

$$Os = \frac{1}{v(\lambda)} \approx \frac{1}{v_0(\lambda)} = \frac{1}{1 - K_2}, \tag{231}$$

and plotted for some values of λ on Fig. 20.

If we assume that $\kappa/\lambda < 1$, we may write

$$Os \approx \frac{1}{1 - K_2} \approx \frac{1}{1 - \frac{1}{\cosh(\kappa)}} = \frac{\cosh(\kappa)}{\cosh(\kappa) - 1}, \tag{232}$$

and if in addition $\kappa < 1$,

$$Os \approx \frac{\cosh(\kappa)}{\cosh(\kappa) - 1} \approx \frac{1 + \kappa^2/2 + ...}{\kappa^2/2 + ...} \approx \frac{2}{\kappa^2} + 1. \tag{233}$$

3.3.4 Epilogue

Lin & Segel [3] conducted this analysis in the seventies. Today we would have started with carrying out some numerical experiments. But since the problem has three dimensionless parameters, it is difficult to obtain a full understanding only by doing this. It is interesting that the straightforward "naive" perturbation expansion breaks down, and that the modified perturbation solution seems to give a very reasonable answer already to leading order (details are left to the reader to check out).

To leading order the emergent osmolarity is only dependent on λ and κ. The perturbation expansion therefore gives an analytical insight that is not so easily obtained by numerical experiments alone.

The dimensionless parameters that we end up with often express important properties of the model we have created. This is the case also in this example, where the parameter κ can be interpreted as follows:

$$\frac{\kappa^2}{2} = \frac{PcC_0L^2}{AD2} = \frac{PcC_0L^2}{AD2}\frac{\bar{C}-C_0}{\bar{C}-C_0} = \frac{\left(cLP(\bar{C}-C_0)\right)C_0\frac{1}{A}}{D\left(\bar{C}-C_0\right)/(L/2)}. \tag{234}$$

($\bar{C}$ is the mean concentration in the channel). The water entering by osmosis is approximately equal to $cLP(\bar{C}-C_0)$, and consequently the convective flux of salt out of the channel is

$$F_{conv.} \approx PcL(\bar{C}-C_0)\times C_0 \times \frac{1}{A}. \tag{235}$$

Similarly, we can write

$$F_{diff} \approx -D\overline{\left(\frac{dC^*}{dx^*}\right)} \approx D\frac{\bar{C}-C_0}{L/2}. \tag{236}$$

Thus,

$$\frac{\kappa^2}{2} \approx \frac{F_{conv.}}{F_{diff.}}, \tag{237}$$

and

$$Os = \frac{Os^*}{C_0} = \frac{F^*(L)}{v^*(L)C_0} = \frac{F_{conv.}+F_{diff}}{F_{conv.}} = 1 + \frac{F_{diff}}{F_{conv.}} \approx 1 + \frac{2}{\kappa^2}, \tag{238}$$

which we also found above.

References

[1] Barenblatt, G.I.: *Scaling, Self-similarity, and Intermediate Asymptotics*, Cambridge Texts in Applied Mathematics, 1996.

[2] Brigham, P.W.: *Dimensional Analysis,* Yale University Press, 1922.

[3] Lin, C.C. and L.A. Segel: *Mathematics Applied to Deterministic Problems in the Natural Sciences*, SIAM Classics in Applied Mathematics, 1988.

[4] Munson, B.R., D.F. Young and T.H. Okiishi: *Fundamentals of Fluid Mechanics,* Wiley 1990.

[5] Sedov, L.I.: *Similarity and Dimensional Methods in Mechanics*, Acad. Press, 1959.

[6] `http://www.cut-the-knot.org/pythagoras/index.shtml`

[7] Taylor, E.S.: *Dimensional Analysis for Engineers,* Clarendon Press, 1974.

4 SELECTED EXERCISES

Hint: Use *Wikipedia* or *Internet* for sorting out basic physical concepts.

4.1 Dimensional Analysis

Exercise 1

State the SI-units for the following physical quantities: (*i*) Acceleration, (*ii*) Mass density, (*iii*) Electrical power, (*iv*) Air pressure, (*v*) Specific heat capacity, (*vi*) Heat conduction coefficient.

Exercise 2

Mechanical stress has the same unit as pressure (Force per unit area). For a *Newtonian fluid* (like water and air) flowing in the x-direction the so-called *shear stress* on a plane parallel to the xy-plane is given by

$$\tau = \mu \frac{\partial u\,(x, y, z)}{\partial y}, \tag{239}$$

where u is the velocity in the x-direction at (x, y, z). What is the unit for the constant μ, called the *dynamic viscosity*?

Exercise 3

Which combinations of *core variables* from the set $\{R_1, \cdots, R_6\}$ may be used if the dimensional matrix is

	R_1	R_2	R_3	R_4	R_5	R_6
F_1	1	1	-1	0	2	2
F_2	-2	-1	1	0	-3	-2
F_3	0	1	0	1	0	2

.

Exercise 4

An open cylindrical tank with diameter, D, is filled to height, h, with a fluid of density, ρ. The bottom has thickness, d, and an elasticity module, E (E is measured in Pascal, like stress). Because of the weight of the fluid, the bottom will sink somewhat, most at the center (No sinkage at the rims). Show that the sinking (distance, δ) in the centre of the bottom may be expressed as

$$\frac{\delta}{D} = \Phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{Dg\rho}\right), \tag{240}$$

where g is the acceleration of gravity.

Exercise 5

A skydiver in *free fall* with speed U experiences a *drag* (friction force) from the surrounding air. The drag may be written as

$$F_d = \frac{1}{2}\rho_{\text{air}} A U^2 \phi\left(\frac{U\sqrt{A}}{\nu}\right), \tag{241}$$

where ρ_{air} is the density of air, A is the cross-sectional area of the skydiver, and ν the kinematic viscosity of the air.

(a) Show how this expression for F_d may be found by dimensional analysis (*Hint*: Use, if necessary, the formula in Eq. 241 first to determine the units of the involved parameters).

(b) After a while the free fall jumper will be falling with constant speed. Find an expression for this speed if we assume that $\phi(x) = 1$.

(*Hint*: The force of gravity, pulling the skydiver downwards, is $F_g = mg$, where m is the skydiver's mass and g the acceleration of gravity. Use that F_g is equal to F_d when the speed is constant).

(c) Estimate the free fall speed in km/hour if we assume that $\phi(x) = 1$.

Exercise 6

An industrial tank holding a chemical liquid has a hole near the bottom. The chemical is flowing through the hole at an amount Q, measured in m^3/s. It is reasonable to assume that Q depends on the diameter of the hole and the pressure difference Δp in the fluid between the inner and outer sides of the hole. In addition, we expect that the flow is governed by the fluid's density ρ and dynamic viscosity μ. Use dimensional analysis to show that the expression for Q under these assumptions may be written

$$Q = \frac{d^2 \Delta p^{1/2}}{\rho^{1/2}} \phi \left(\frac{d \rho^{1/2} \Delta p^{1/2}}{\mu} \right), \tag{242}$$

where ϕ is an unknown function of only one variable.

Exercise 7

The force (F) on an aircraft propeller depends on its diameter, d, the speed of the airplane, U, the density of the air, ρ, the number of rotations per second, ω, and the viscosity of the air, μ.

Show how dimensional analysis is used to find the formula

$$F = \rho U^2 d^2 \phi \left(\frac{\omega d}{U}, \frac{U d}{\mu / \rho} \right), \tag{243}$$

where ϕ is an unknown function in two variables (Hint: Use, if necessary, Eq. 243 to find the units for the variables).

Exercise 8

By measuring the pressure drop p in a tube vs. the time t it took to fill a cup with volume V, *Bose*, *Bose* and *Ruert* around 1910 found the relations on Fig. 21 (left) for water, chloroform, bromoform and mercury. Show, by introducing dimensionless variables (using the density ρ and viscosity μ, $[\mu] = \mathrm{kg\ s^{-1} m^{-1}}$), that it exists one common relation covering all the cases. That is, find the variables along the axes in *von Kármán's* alternative presentation of the same data, as shown in the figure to the right.

Exercise 9

We consider an elastic rubber band which may be stretched many times its original length l_0. The rubber band has a "density" ρ which we measure in mass per unit length, that is, kg/m. How does the density ρ vary when we stretch the band to a length l from its original length l_0 and density ρ_0? After stretching the band more that twice its original length, we pluck the band like a guitar string. This experiment shows, somewhat unexpected, that the *pitch* (=frequency ω) remains almost constant when we vary the length (*try it*!). However, when stretching the band up towards its breaking limit, the frequency increases somewhat. The force F required for stretching the band to length l is proportional to $l - l_0$ over most of the range, that is, $F = F_0 \frac{l - l_0}{l_0}$, where F_0 is a constant. Use dimensional analysis to explain the behaviour of the frequency. (*Hint*: Assume

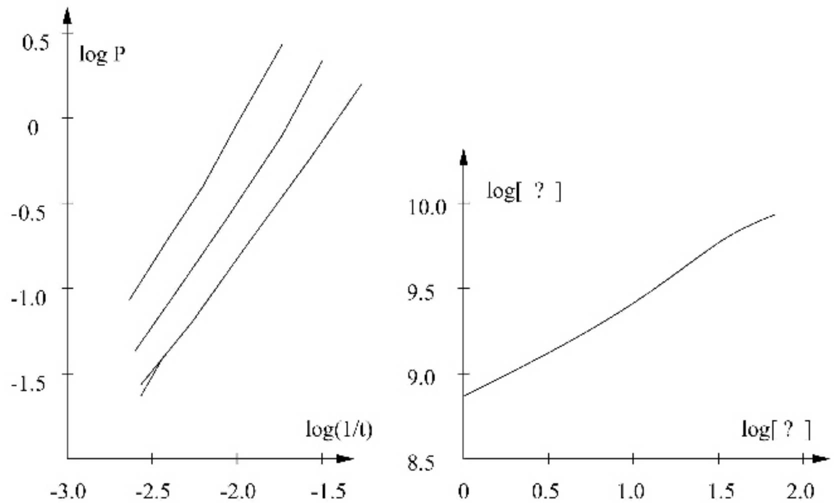

Figure 21: Presentation of the data in the original paper (left) and vonKarman's revised graph after applying dimensional analysis (right).

first that $\omega = f(l, \rho, F)$, apply dimensional analysis, and then introduce the expression of the density as a function of l, l_0 and ρ_0).

Exercise 10

In forest assessment one wants to estimate the volume V (also called the *cubic content*) of a tree by measuring its height (h) and diameter (d) at the root. A test example in *Minitab* suggests the following formula (based on multilinear regression) for American cherry trees:

$$V^{1/3} = \beta_0 + \beta_1 d + \beta_2 h + \beta_3 d^2. \tag{244}$$

Here $\{\beta_i\}$ are regression coefficients calculated from a set of calibration data.

(a) Americans use *foot* and most of the rest of the world *meter* as the basic length unit. Is it possible to use the same values for $\{\beta_i\}$ in both cases?

(b) Show that dimensional analysis, based on the variables v, d, and h, instead recommends applying a relationship of the form

$$\pi_1 = \phi(\pi_2). \tag{245}$$

Find π_1 and π_2, and give examples of what the function ϕ could be for some "idealized trees", e.g. cylinders and cones.

(For students who know *Minitab*, it might be interesting to check which of the models are best: The model in Eq. 244, or a regression model based on 245. The calibration data used in Minitab 14 are stored in the MTW file `trees.mtw`)

Exercise 11

The necessary force (F) to keep a ship at a constant speed (U) depends on its shape; primarily the length (L), width (W), and its depth into the water (D). In addition, the water density, ρ, the viscosity, ν, and the acceleration of gravity, g, play a part.

Use dimensional analysis to find an expression for the force which includes the two most famous dimensionless numbers in ship design:

$$\textit{Froude number}: \ Fr = U/\sqrt{Lg}, \tag{246}$$
$$\textit{Reynolds number}: \ Re = LU/\nu. \tag{247}$$

Ideally, a scale model[1)] of the ship should be tested experimentally in water by keeping the dimensionless numbers for the model equal to those of the original ship. Is this really possible?

(*Hints*: $[F] = \text{kgm/s}^2$, $[\rho] = \text{kg/m}^3$, $[\nu] = \text{m}^2\text{/s}$, $[g] = \text{m/s}^2$).

[1)] A *scale model* is a model of the ship with the same geometric shape, but with a smaller size (Say, L =1m for the model, compared to 200m for the original ship).

4.2 Scaling and Regular Perturbation

Exercise 1

State what it means to scale (i) *a physical quantity*, (ii) *an equation.*

Exercise 2

The following expressions have been proposed as the time scale for the function $u^*(t^*) = A\cos(2\pi f_0 t^*)$:

$$\begin{aligned} T &= 1/f_0, \\ T &= 1/(2\pi f_0), \\ T &= 1/(\pi f_0), \\ T &= 500\pi f_0^{-1}. \end{aligned} \tag{248}$$

May all these be used as the time scale?

Exercise 3

A common mathematical model for the size of a population $y^*(t^*)$ as a function of time t^* is described by the *logistic equation*

$$\frac{dy^*}{dt^*} = ry^*\left(1 - \frac{y^*}{K}\right). \tag{249}$$

Here r is called the *growth rate* and K the *sustainable capacity.*

(a) Which scale is suitable for y^*?

(b) Determine a time scale when $y^* \ll K$.

(c) Introduce these scales into the equation so that it becomes dimensionless (The equation can easily be solved by inserting $y = 1/u$ and solving for u).

Exercise 4

In many dynamic systems one talks about *time constants*. For an exponential function, $u(t) = A\exp(-at)$, the *time constant* is defined as follows: First draw the tangent to $u(t)$ at t_0. This tangent is crossing the x-axis at t_1, and the time constant is defined $T = |t_1 - t_0|$. Show that this definition also follows from the *rule of thumb*,

$$T = \frac{\max|u(t)|}{\max|du(t)/dt|}. \tag{250}$$

Exercise 5: Case B in the discussion of the falling sphere in a fluid led to the equation

$$2\ddot{x} + \varepsilon\dot{x} = 1,\ x(0) = 0,\ \dot{x}(0) = 0,\ 0 < \varepsilon \ll 1. \tag{251}$$

This equation has the exact solution

$$x_{\text{sol}}(t) = \frac{2}{\varepsilon^2}\left(e^{-\varepsilon t/2} - 1\right) + \frac{t}{\varepsilon}. \tag{252}$$

(a) Determine x_0, x_1 and x_2 in the regular perturbation expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots, \tag{253}$$

and show that it agrees with the start of the power series development in ε of the exact solution.

(b) An approximate solution $x_a(t,\varepsilon)$ is a *uniform approximation* to the exact solution, x_{sol}, on the interval $[0,1]$ if

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,1]}\left|x_a(t,\varepsilon) - x_{\text{sol}}(t)\right|\right) = 0. \tag{254}$$

Does this apply to $x_a(x,\varepsilon) = x_0(t) + \varepsilon x_1(t)$? What if we replace $[0,1]$ with$[0,\infty)$?

Exercise 6

Consider the problem

$$y''(t) + \varepsilon y'(t) + 1 = 0, \tag{255}$$

$$y(0) = 0\ ,\ y'(0) = 0,\quad 0 < \varepsilon \ll 1. \tag{256}$$

Determine the start of the perturbation expansion $y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)$ to the solution for $t \geq 0$. Compare to the exact solution. (*Hint*: The general solution of Eq. 255 has the general form $y(t) = A + Be^{-\varepsilon t} - t/\varepsilon$)

Exercise 7

This problem is somewhat similar to the sphere falling in a fluid (the scaling model problem without gravity), but in this case the friction is more realistic and nonlinear. The equation may be written

$$m\frac{dv^*}{dt^*} = -av^* + bv^{*2},\quad v^*(0) = V_0. \tag{257}$$

and

$$v^*(0) = V_0. \tag{258}$$

We have been told that $a, b > 0$, and also that $bV_0 \ll a$.

(a) Find the (obvious) scale for v^* and then the scale for time, T, from the simplified equation, $m\frac{dv^*}{dt^*} = -av^*$, by the *rule of thumb*

$$T = \frac{\max\left|v^*(t)\right|}{\max\left|dv^*(t)/dt^*\right|}. \tag{259}$$

Show that this scaling leads to the equation

$$\frac{dv}{dt} = -v + \varepsilon v^2, \tag{260}$$

$$v(0) = 1,\ \varepsilon \ll 1. \tag{261}$$

(b) Determine v_0 and v_1 in the series expansion $v(t) = v_0(t) + \varepsilon v_1(t) + \cdots$. Is this result reasonable for all $t > 0$ when the general solution of $\dot{y} + y - \varepsilon y^2 = 0$ is

$$y(t) = \frac{e^{-t}}{C + \varepsilon e^{-t}}? \tag{262}$$

Exercise 8

During the modeling of the sprinters, we derived the equation

$$M\frac{du^*}{dt^*} = Mp^*(t^*) - M\frac{u^*}{\tau} - F_{\text{air}}, \tag{263}$$

where M is the runner's mass, u^* the velocity, p^* a "performance variable", τ a time constant, and F_{air} the air resistance.

(a) Explain why the term for air resistance, found from dimensional analysis, ought to be stated as

$$F_{\text{air}} = \tfrac{1}{2}\rho_{\text{air}} C_D(Re) A(u^* - W)\left|u^* - W\right|. \tag{264}$$

(Here ρ_{air} is the air density, Re the Reynolds number and A the runner's cross-sectional area).

After scaling,

$$p^* = Pp, \tag{265}$$

$$t^* = \tau t, \tag{266}$$

$$u^* = (P\tau)\,u, \tag{267}$$

and without wind, the equation becomes

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t), \tag{268}$$

$$u(0) = 0, \tag{269}$$

where

$$\varepsilon = \tfrac{1}{2}\rho_l C_D(Re)\,\tau^2 P \tfrac{A}{M}. \tag{270}$$

(b) Estimate ε for Usain Bolt and Florence Griffith-Joyner when we assume here and below that $\rho_{\text{air}} = 1.2\text{kg/m}^3$, $C_D(Re) \equiv 1$, $P = 10\text{m/s}^2$ and $\tau = 1\text{s}$.

(c) Verify that the solution of Eq. 268 with $p(t) \equiv 1$ will be

$$u(t) = 1 - e^{-t} + \varepsilon[-1 + 2te^{-t} + e^{-2t}] + \mathcal{O}\left(\varepsilon^2\right). \tag{271}$$

(d) We scale the wind speed in the same way as u^*, so that

$$u^* - W = (P\tau)(u - \delta). \tag{272}$$

Show that the dimensionless maximum speed $u_{\max}$ (when $u_{\max} > \delta$) is given by

$$u_{\max} + \varepsilon(u_{\max} - \delta)^2 = 1. \tag{273}$$

(e) Show that in order to determine the more exact time spent on the running, T_a, on the basis of this model it will be necessary to solve the equation

$$\frac{L}{P\tau^2} = \int_0^{T_a/\tau} u(t)\,dt, \tag{274}$$

where $L = 100$m.

(f) The discussion in the last few paragraphs of the course note is rough, since it is assumed that the runner holds a maximum speed of during the whole distance. The advantage of the tailwind is therefore estimated too large. From the information about her time under controlled conditions, it is possible derive the size of ε, provided that the model holds.

So what is the conclusion of this study? *Tailwind* or *doping*?

Matlab code for those interested (This code may need some modifications for Octave):

```
% SCRIPT
global EPS DELTA
% Parameters:
P = 10;    % Maximum performance factor [m/s^2]
tau = 1.0; % Relaxation time [s]
M = 60;    % Body mass [kg]
A = 0.40;  % Cross sectional area [m^2]
rho = 1.2; % Density [kg/m^3]
W = 0 ;    % Wind speed [m/s]
Cd = 1;    % Drag coefficient
%
EPS = 0.5*rho*Cd*tau^2*P*A/M
DELTA = W/(P*tau)
%
Treal = 0:0.1:12; % NB! Time in seconds
tspan = Treal/tau;
% ODE solver
[t,y] = ode45(@FJfunc,tspan,[0 0]');
T = t*tau;        % Real time
L = P*tau^2*y;    % Real distance
plot(T,L); xlabel('Time [s]') ; ylabel('Distance [m]');
legend('Position','Velocity')
grid

function dydt = FJfunc(t,y)
global EPS DELTA
dydt    = [0 0]';
dydt(1) = y(2);
dydt(2) = 1-y(2)-EPS*(y(2)-DELTA)*abs(y(2)-DELTA);
```