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EXAM IN TMA4195 Mathematical Modeling

English
Saturday, December 10, 2011
9:00 – 13:00

Aids (code C): Approved calculator
Rottman: *Matematisk formelsamling*

Results: January 10, 2012

Problem 1 Write down the logistic equation and give an example where you explain what the equation can model and the meaning of the parameters.

Explain what kind of population models lead to the following scaled system of equations:

$$\frac{dx}{dt} = x(1-y),$$
$$\frac{dy}{dt} = \alpha y(-1+x),$$

for some $\alpha > 0$.

This system has equilibrium points $(0,0)$ and $(1,1)$. Use, if you can, linear stability analysis (linearization) to determine the stability of these equilibrium points.

Problem 2 When a drop of liquid hits a liquid surface, a crown formation appears – see the figure on the right. The number of points N on the crown is assumed to depend on the radius r and impact speed U of the drop, and the density ρ and the surface tension σ of the liquid.

Use dimensional analysis to find a relation for N in terms of r, ρ, U, and σ.

Hint: $[\sigma] = \text{force/length}$.

Problem 3 We are given the following intitial value problem,

$$
(1) \qquad \begin{cases} \ddot{x} = -\epsilon \dot{x}\sqrt{\dot{x}^2+\dot{z}^2}, & t>0; \quad x(0)=0, \quad \dot{x}(0)=1, \\ \ddot{z} = -1-\epsilon \dot{z}\sqrt{\dot{x}^2+\dot{z}^2}, & t>0; \quad z(0)=0, \quad \dot{z}(0)=1, \end{cases}
$$

where $\epsilon > 0$ and $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, etc.

a) Find the initial value problems satisfied by x_0, z_0, x_1 and z_1 in the perturbation expansions

$$x(t) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2) \quad \text{and} \quad z(t) = z_0(t) + \epsilon z_1(t) + O(\epsilon^2),$$

when $x(t)$ and $z(t)$ are the solutions of the initial value problem (1).

Determine x_0 and z_0.

A point projectile travels through a constant gravity atmosphere ($\tilde{z} > 0$) until it hits the ground ($\tilde{z} = 0$). Here $\tilde{x}$ and $\tilde{z}$ denote the horizontal and vertical coordinates of the projectile path. The projectile has mass m and is only affected by two forces: gravity and air resistance

$$\vec{F}_r = -c\vec{v}|\vec{v}|,$$

where $c > 0$ is a constant and $\vec{v}$ is the velocity of projectile. The gravitational acceleration g is constant, and the projectile starts at $t = 0$ at the point $(0, 0)$ with velocity vector (U_0, U_0).

b) Find an initial value problem for the position $(\tilde{x}(\tilde{t}), \tilde{z}(\tilde{t}))$ of the projectile at time $\tilde{t}$.

Find the good scaling for the problem when gravity is the dominating force, and show that the initial value problem under this scaling is (1).

Hint: Remember that $\tilde{z} \geq 0$. What is the natural velocity scale in the x and z directions?

Problem 4 Two chemical substances A and B react with rate constant $k > 0$ to form $2A$:

$$(2) \qquad A + B \xrightarrow{k} 2A.$$

We let $\tilde{a}$ and $\tilde{b}$ denote the concentrations of A and B, and assume that $\tilde{a}(0) = a_0$ and $\tilde{b}(0) = b_0$.

a) In a well-stirred tank reactor, $\tilde{a}$ and $\tilde{b}$ are uniform in space and vary only in time. Show that

$$\frac{d\tilde{a}}{d\tilde{t}} = k\tilde{a}(a_0 + b_0 - \tilde{a}), \quad \tilde{t} > 0.$$

Hint: The law of mass action. Show that $\tilde{a}(\tilde{t}) + \tilde{b}(\tilde{t}) = \text{constant}$.

The reaction (2) takes place in an unstirred long thin tube reactor. In a simple model we assume that the reactor is the $\tilde{x}$-axis, and since it is not stirred, $\tilde{a}$ and $\tilde{b}$ may vary in $\tilde{x}$ and $\tilde{t}$. We also assume that $a_0(\tilde{x})$ is not constant while

$$a_0(\tilde{x}) + b_0(\tilde{x}) = M = \text{constant}.$$

Since the concentrations depend on $\tilde{x}$, there will be diffusion of reactants in addition to reaction. The diffusion coefficient D is assumed to be constant.

b) State Fick's law for diffusion, and write down the conservation law for the substance A in the "control volume" $I = [c, d]$.

Derive the conservation law in differential form, and show that there is a scaling such that it takes the form

$$\frac{\partial a}{\partial t} = \frac{\partial^2 a}{\partial x^2} + a(1-a), \quad t > 0, \ x \in \mathbb{R}. \tag{3}$$

Equation (3) has two constant equilibrium solutions $a = 0$ and $a = 1$. To check the stability of these solutions, we must study the behaviour of solutions that are close at $t = 0$.

c) Let $a = 1 + c$ be a solution of (3) such that $c(x, 0) = c_0(x)$ is bounded.

Find the solution c_L of the linearized equation for c and compute $\lim_{t\to\infty} c_L(x, t)$.

Is the solution $a = 1$ of (3) stable or unstable according to linear stability analysis (linearization)?

Hint: The fundamental solution of the heat equation, $c_F(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$, satisfies $c_F \geq 0$ and $\int_{-\infty}^{\infty} c_F(x,t)dx = 1$.

Problem 5 In a scaled fluid dynamics model of car traffic along a one-way road, the density of cars ρ satisfies the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} j(\rho) = 0,$$

where $j(\rho) = \rho(1-\rho)$ is the flux of cars. For all $t > 0$, there is a red traffic light at $x = 0$, while

$$\rho(x, 0) = \frac{1}{4} \quad \text{for all} \quad x \in \mathbb{R}.$$

Find the future car density to the left of the traffic light, i.e. find ρ for all $x < 0$ and $t > 0$.