TMA4195 Mathematical modeling 2011

Suggested solution exam 2011

Problem 1:
The logistic equation models population growth:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right),$$

where N is the size of the population, r the growth rate, and K the carrying capacity.

The Lotka–Volterra system models predator (y) and prey (x) populations and how they interact.

Linear stability analysis:

$$f = \begin{bmatrix} x(1-y) \\ \alpha y(-1+x) \end{bmatrix} \implies Df = \begin{bmatrix} 1-y & -x \\ \alpha y & \alpha(-1+x) \end{bmatrix}.$$

We observe that $Df(0,0)$ has eigenvalues $\lambda_1 = 1, \lambda_2 = -\alpha$, so since $\max_{i=1,2} \operatorname{Re}\lambda_i > 0$, $(0,0)$ is an unstable equilibrium point. The matrix $Df(1,1)$ has eigenvalues $\lambda_i = \pm \mathrm{i}\sqrt{\alpha}$, so $\max_{i=1,2} \operatorname{Re}\lambda_i = 0$ and we get no conclusion from linear stability analysis.

Problem 2:
The dimension matrix A is

	r	ρ	U	σ	N
m	1	−3	1	0	0
s	0	0	−1	−2	0
kg	0	1	0	1	0

Since $\operatorname{rank} A = 3$, we get $5-3 = 2$ dimensionless combinations. Trial and error quickly leads to

$$\pi_1 = N, \quad \text{and} \quad \pi_2 = \frac{\rho}{\sigma}U^2 r.$$

If there is a relation $\Phi(N, r, \rho, U, \sigma) = 0$, Buckingham's π-theorem tells us that there is an equivalent dimensionally consistent relation $\Psi(\pi_1, \pi_2) = 0$. Solving for N, we find that

$$N = \tilde{\Psi}(\pi_2) = \tilde{\Psi}\left(\frac{\rho r U^2}{\sigma}\right),$$

for some unknown function $\tilde{\Psi}$.

Problem 3:

a) Simply insert the perturbation expansions into the initial value problem and equate terms of equal order in ε. This leads to the following four initial value problems:

$$\begin{aligned} \ddot{x}_0 &= 0, & x_0(0) &= 0, \dot{x}_0(0) = 1, \\ \ddot{x}_1 &= -\dot{x}_0\sqrt{\dot{x}_0^2+\dot{x}_0^2}, & x_1(0) &= \dot{x}_1(0) = 0, \\ \ddot{z}_0 &= -1, & z_0(0) &= 0, \dot{z}_0(0) = 1, \\ \ddot{z}_1 &= -\dot{z}_0\sqrt{\dot{x}_0^2+\dot{x}_0^2}, & z_1(0) &= \dot{z}_1(0) = 0. \end{aligned}$$

We solve for x_0 and z_0 and get

$$x_0 = t, \qquad z_0 = -\tfrac{1}{2}t^2 + t.$$

b) Since $\vec{v} = (\dot{\tilde{x}}, \dot{\tilde{z}})$, Newton's second law gives us

$$m\begin{bmatrix}\ddot{\tilde{x}}\\ \ddot{\tilde{z}}\end{bmatrix} = \vec{F}_g + \vec{F}_r = -mg\begin{bmatrix}0\\1\end{bmatrix} - c\begin{bmatrix}\dot{\tilde{x}}\\ \dot{\tilde{z}}\end{bmatrix}\sqrt{\dot{\tilde{x}}^2+\dot{\tilde{z}}^2},$$

with inital conditions $\big(\tilde{x}(0), \tilde{z}(0)\big) = (0,0)$, $\big(\dot{\tilde{x}}(0), \dot{\tilde{z}}(0)\big) = (U_0, U_0)$.

We use the scaling

$$\tilde{x} = Xx, \quad \tilde{z} = Zz, \quad \tilde{t} = Tt.$$

Observe that $\max|\dot{\tilde{x}}| = \max|\dot{\tilde{z}}| = U_0$, so it is natural to set

$$U_0 = \frac{X}{T} = \frac{Z}{T}.$$

Inserting this into the initial value problem, we get

$$m\frac{X}{T^2}\begin{bmatrix}\ddot{x}\\ \ddot{z}\end{bmatrix} = -mg\begin{bmatrix}0\\1\end{bmatrix} - c\frac{X^2}{T^2}\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix}\sqrt{\dot{x}^2+\dot{z}^2},$$

Gravity dominates, so we balance the first and second terms, giving $X = gT^2 = U_0^2/g$. Inserting this, we end up with the equations from the text, with

$$\varepsilon = \frac{cU_0^2}{mg}.$$

Problem 4:

a) The law of mass action gives us that the reaction rate $r = k\tilde{a}\tilde{b}$. Each reaction produces one molecule of substance A and removes one molecule of substance B, so

$$\frac{\mathrm{d}\tilde{a}}{\mathrm{d}\tilde{t}} = r = k\tilde{a}\tilde{b}, \qquad \frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tilde{t}} = -r = -k\tilde{a}\tilde{b}.$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}(\tilde{a}+\tilde{b}) = 0 \qquad \Longrightarrow \qquad \tilde{a}+\tilde{b} = a_0 + b_0,$$

and hence

$$\frac{\mathrm{d}\tilde{a}}{\mathrm{d}\tilde{t}} = k\tilde{a}(a_0 + b_0 - \tilde{a}).$$

b) Fick's law states that the diffusive flux of a substance with concentration c is $\vec{j}_c = -D\nabla c$.

The conservation law for substance A in $I = [c, d]$ is then

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_I \tilde{a}\,\mathrm{d}x = -\big(j_a(d) - j_a(c)\big) + \int_I k\tilde{a}(M-\tilde{a})\,\mathrm{d}x.$$

We can transform the integral form to differential form by the standard procedure of setting $d = c + \Delta x$, dividing by Δx and letting Δx tend to 0. This leads to the PDE

$$\tilde{a}_{\tilde{t}} = D\tilde{a}_{\tilde{x}\tilde{x}} + k\tilde{a}(M-\tilde{a}).$$

If we choose scales $\tilde{a} = Aa$, $\tilde{x} = Xx$, $\tilde{t} = Tt$ with

$$A = M, \qquad T = \frac{1}{Mk}, \qquad X = \sqrt{\frac{D}{Mk}},$$

and divide by A/T, we end up with the equation in the text.

c) If we insert $a = 1 + c$ into the scaled PDE and linearize around $c = 0$, we get

$$(c_L)_t = (c_L)_{xx} - c_L.$$

Letting $\tilde{c} = \mathrm{e}^t c_L$, the equation is reduced to the heat equation

$$\tilde{c}_t = \tilde{c}_{xx},$$

which is solved by convolution with the fundamental solution c_F:

$$\tilde{c} = c_F * \tilde{c}_0 = \int_{-\infty}^{\infty} c_F(x-y,t)\tilde{c}(y,0)\,\mathrm{d}y.$$

Going back, we get

$$c_L(x,t) = \mathrm{e}^{-t}\tilde{c}(x,t) = \mathrm{e}^{-t}\int_{-\infty}^{\infty} c_F(x-y,t)\,c_L(y,0)\,\mathrm{d}y.$$

Using the hint, we calculate

$$|c_L| \le \mathrm{e}^{-t}\int_{-\infty}^{\infty} c_F(x-y,t)|c_L(y,0)|\,\mathrm{d}y \le \mathrm{e}^{-t}\max|c_L(y,0)|\cdot 1 \underset{t\to\infty}{\longrightarrow} 0.$$

From this, we conclude that all small perturbations of $a=1$ die out in time, and $a=1$ is an asymptotically stable equilibrium solution.

Problem 5:
The red light at $x=0,\ t>0$ implies that the flux $j(\rho)=0$ at $x=0,\ t>0$, i.e. $\rho=0$ or $\rho=1$ at $x=0,\ t>0$. We must choose $\rho=0$ or $\rho=1$ at $x=0$ so that the characteristics go into the domain $x<0$. Since the kinematic velocity is

$$j'(\rho) = 1-2\rho = \begin{cases} 1, & \rho=0 \\ -1, & \rho=1, \end{cases}$$

we must choose $\rho=1$ at $x=0$. At time $t=0$, we are given $\rho=1/4$, so $j'(\rho)=1/2>0$. Thus, the characteristics cross and we get a shock solution

$$\rho(x,t) = \begin{cases} 1, & S(t)\le x\le 0 \\ 1/4 & x\le S(t), \end{cases}$$

where the schock curve $S(t)$ satisfies the Rankine–Hugoniot condition

$$\dot{S}(t) = \frac{j(\rho_{\text{left}})-j(\rho_{\text{right}})}{\rho_{\text{left}}-\rho_{\text{right}}} = \frac{j(1/4)-j(1)}{1/4-1} = -1/4, \quad S(0)=0,$$

so $S(t)=-t/4$.