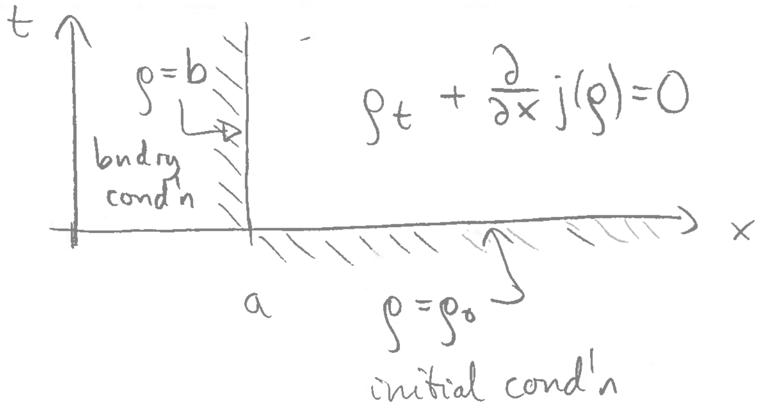
Matmod 12.10.12

A. Boundary value problem

$$(1)\begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0 , & x > a,\ t > 0 \\ \rho(a,t) = b , & x = a,\ t > 0 \\ \rho(x,0) = \rho_0(x) , & x > a,\ t = 0 \end{cases}$$



<u>Fact:</u> Char.'s <u>can</u> start on bndry!

<u>Char. eq'ns:</u> $z(t) = \rho(x(t), t)$

$$\dot{x} = j'(z) = c(z) ; \quad \dot{z} = 0$$

Init. cond'ns

$$x(t_0) = x_0 ; \quad z(t_0) = \rho(x(t_0), t_0) = \rho(x_0, t_0)$$

Sol'n

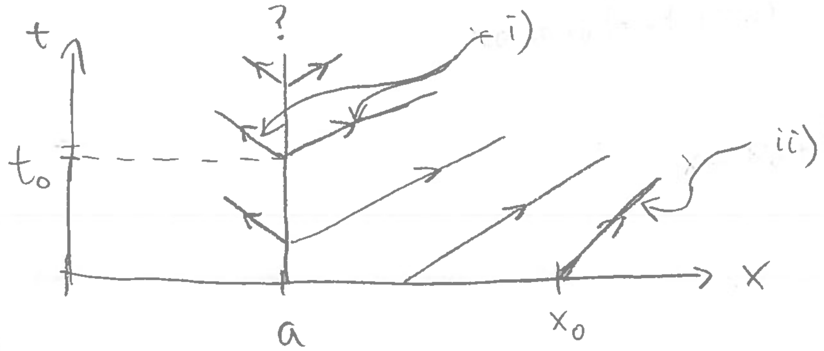
$$\begin{cases} x = x_0 + (t - t_0)\, c(\rho(x_0, t_0)) \\ z = \rho(x_0, t_0) \end{cases}$$

i) $x_0 > a$, $t_0 = 0$ $\left(\rho(x_0,t_0) = \rho_0(x_0)\right)$:

$$\begin{cases} x = x_0 + t\, c(\rho_0(x_0)) \\ z = \rho_0(x_0) \end{cases}$$

ii) $x_0 = a$, $t_0 > 0$ $\left(\rho(x_0,t_0) = b\right)$:

$$\begin{cases} x = a + (t - t_0)\, c(b) \\ z = b \end{cases}$$



<u>2 possibilities:</u>

1.) <u>Inflow</u>: $c(b) = j'(b) \geq 0$

Char's at $x = a$ go <u>into</u> domain $x \geq a$

$\Longrightarrow$ sol'ns found as before:

meth. of char. + shocks + rarefaction

2.) <u>Outflow</u>: $c(b) = j'(b) < 0$

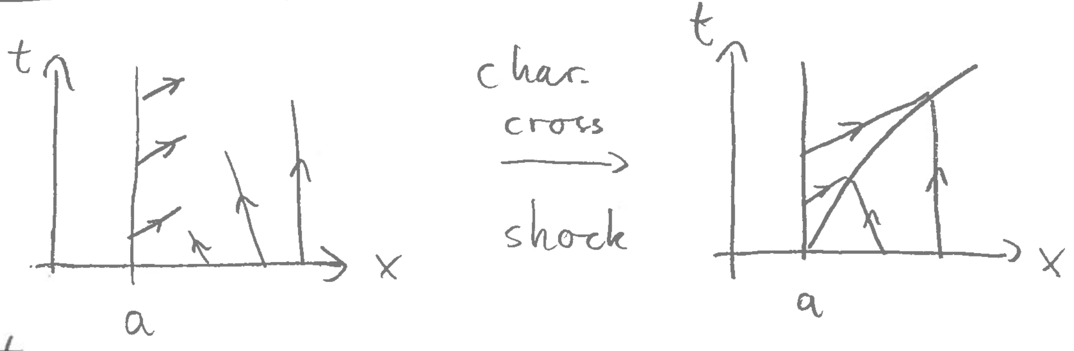
Char's at $x = a$ <u>leave</u> domain $x \geq a$

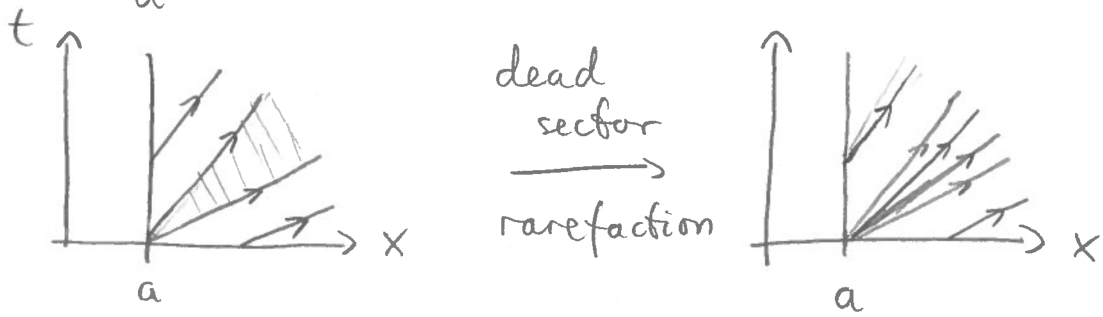
$\Longrightarrow$ sol'n does <u>not see</u> bndry. cond'n at $x = a$:

meth. of char. + shocks + rarefaction

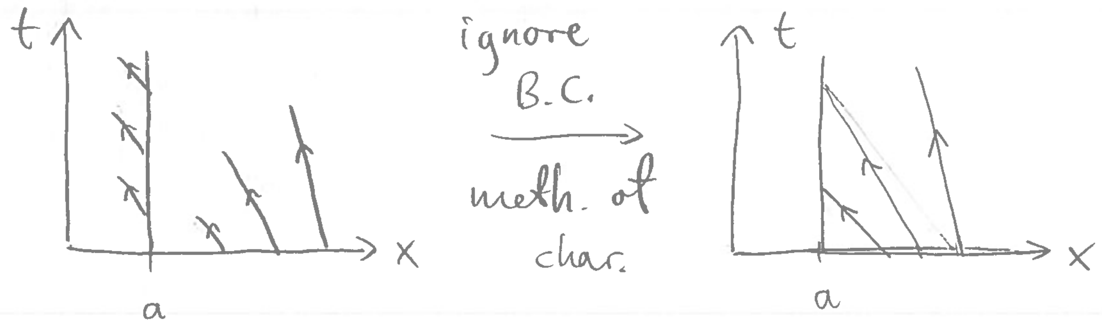
+ <u>ignore</u> B.C.

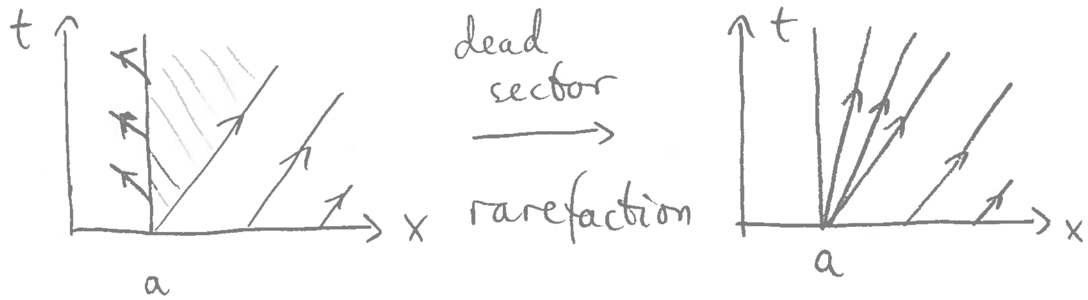
Ex. 1: Inflow





Ex. 2: Outflow





Obs. 1:

i) Bndry cond'ns (B.C.s) can <u>only</u> be imposed at inflow bndries

ii) In Ex. 2, typically $\rho(a^+, t) \neq b$!

B. Flux condition

$$(2)\begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0, & x > a,\ t > 0 \\ j(\rho) = b, & x = a,\ t > 0 \\ \rho(x,0) = \rho_0(x), & x > a,\ t = 0 \end{cases}$$

Obs 2: $j = 0$ at $x = a$

$\Rightarrow$ no flux (transport) across $x = a$

Idea: Convert (2) to (1) by solving

$$(3) \qquad j(\rho) = b \quad \text{for } \rho.$$

Obs. 3: If (3) has many sol'ns, select only sol'ns giving inflow bndry! (Why?)

C. Traffic – red light

Car density ρ:

$$\rho_t + \frac{\partial}{\partial x} j(\rho) = 0$$

where $j(\rho) = \rho(1-\rho)$

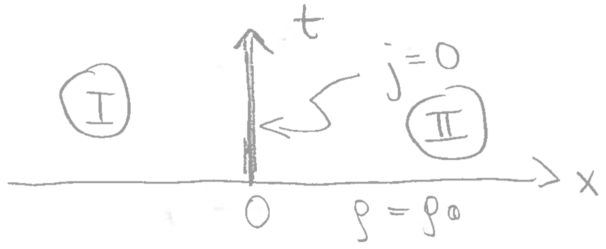
Traffic light at $x = 0$:

$t < 0$ green, $t \geq 0$ red

$t \le 0: \ \rho(x,t) \equiv \frac{1}{2}$ (unif. traffic)

<u>Sol'n for $t>0$:</u>

Red light $\Longrightarrow$ no flux at $x = 0$!



2 bndry value problems: In I ($x<0$) and in II ($x>0$)!

Bndry cond'n:

$$0 = j(\rho) = \rho(1-\rho) \Longrightarrow \rho = 0 \text{ or } \rho = 1$$

Choose <u>inflow</u> values:

$$c(\rho) = 1 - 2\rho = \begin{cases} 1 & , \ \rho = 0 \\ -1 & , \ \rho = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} \rho = 1 \ (\rho = 0) \text{ gives inflow (outflow) in I} \\ \rho = 0 \ (\rho = 1) \text{ ——— '' ——— II} \end{cases}$$

(reasonable choices)

<u>Sol'n in I:</u>

$$\begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0, & x < 0, \ t > 0 \\ \rho = 1, & x = 0, \ t > 0 \\ \rho = \frac{1}{2}, & x < 0, \ t = 0 \end{cases}$$

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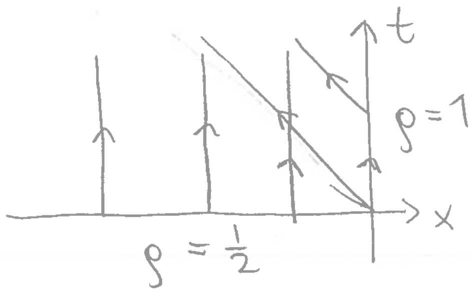
Meth. of char's

$$x = x_0 + (t-t_0)\, c(\rho(x_0,t_0))$$

$$= \begin{cases} x_0 + t\, \underset{\overset{\shortparallel}{0}}{c(\tfrac{1}{2})} = x_0 & , t_0=0,\ x_0 \le 0 \\ 0 + (t-t_0)\underset{\overset{\shortparallel}{-1}}{c(1)} = t_0 - t & , t_0 > 0,\ x_0 = 0 \end{cases}$$



Char's collide ⟹ shock

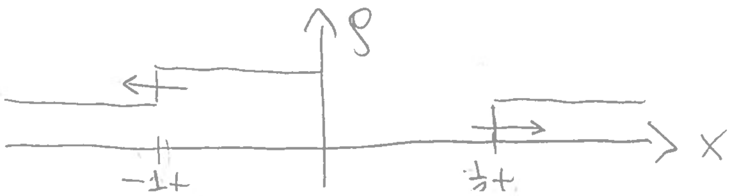
$$\dot{s} = \frac{j(\tfrac{1}{2}) - \overset{\overset{0}{\shortparallel}}{j(1)}}{\tfrac{1}{2} - 1} = -\frac{1}{2} \ , \ s(0) = 0$$

$$\overset{\text{chk.}}{\Longrightarrow} \rho(x,t) = \begin{cases} 1 & , -\tfrac{1}{2}t < x < 0 \\ \tfrac{1}{2} & , x < -\tfrac{1}{2}t \end{cases}$$

<u>Sol'n in II:</u>

$$\begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0 & , \quad x > 0,\ t > 0 \\ \rho = 0 & , \quad x = 0,\ t > 0 \\ \rho = \tfrac{1}{2} & , \quad x > 0,\ t = 0 \end{cases}$$

$$\overset{\text{chk.}}{\rightsquigarrow} \rho(x,t) = \begin{cases} 0 & , \ 0 \le x \le \tfrac{1}{2}t \\ \tfrac{1}{2} & , \ \tfrac{1}{2}t \le x \end{cases}$$



(reasonable?)

D. Traffic – 2 → 1 lanes

2 → 1 lanes at $x = 0$

$$\Rightarrow v(\rho) = \begin{cases} 1 - \rho & x < 0 \\ 1 - \frac{\rho}{\frac{1}{2}} & x > 0 \end{cases}$$

max density in 1 lane road

$$\Rightarrow j(\rho) = \rho v(\rho) = \begin{cases} \rho(1-\rho) = j_-(\rho) & x < 0 \\ \rho(1-2\rho) = j_+(\rho) & x > 0 \end{cases}$$

Cond'n at $x=0$: Flux continuity

$$j_-(\rho(0^-,t)) = j_+(\rho(0^+,t)) \quad t > 0$$

[Conservation – if not cars created/destroyed at $x=0$]
(follows from cons. law in int. form)

<u>Obs. 4:</u> j <u>discont.</u> in ρ but $j \circ \rho$ <u>cont.</u> in x!

($\Rightarrow \rho$ discont. at $x=0$)

Morning rush reach $x=0$ at $t=0$

$$\rho(x,0) = \begin{cases} \frac{1}{2} & , \ x < 0 \\ 0 & , \ x > 0 \end{cases}$$

Obs: $j_-(\rho(0^-,0)) = \frac{1}{4} > \max_\rho j_+(\rho) = \frac{1}{8}$

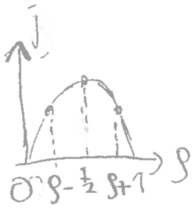
What is flux at $x = 0^+$?

– <u>Assume</u> $j_+(\rho(0^+,t)) = \text{maximal} = \frac{1}{8}$

(modeling assumption!)

$$\underset{\text{flux conf.}}{\Longrightarrow} \begin{cases} x = 0^- : & j_- = \frac{1}{8} \Rightarrow \rho_\pm = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{2}}\right) \\ x = 0^+ : & j_+ = \frac{1}{8} \Rightarrow \rho = \frac{1}{4} \end{cases}$$

Obs. 5:



i) $\rho_- < \frac{1}{2} < \rho_+$

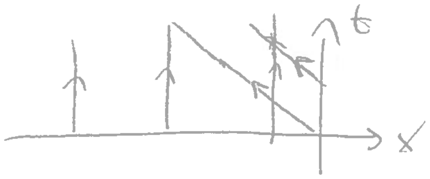
ii) $c_-(\rho_-) > 0,\ c_-(\rho_+) < 0,\ c_+(\frac{1}{4}) = 0$

Sol'n on $x<0$:

$$\begin{cases} \rho_t + \frac{\partial}{\partial x} j_-(\rho) = 0, & x<0, t>0 \\ \rho = \rho^+, & x = 0, t>0 \\ \rho = \frac{1}{2}, & x<0, t=0 \end{cases}$$

Meth. of char's:

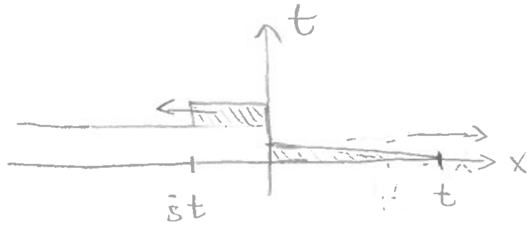
$$x = \begin{cases} x_0 + t\, \overset{0}{\overset{\shortparallel}{c_-(\frac{1}{2})}} & x_0 < 0, t>0 \\ 0 + (t-t_0)\, \underset{\overset{\wedge}{0}}{c_-(\rho_+)} & x_0 = 0, t > t_0 \end{cases}$$



Char. collide $\Rightarrow$ shock

$$\dot{s} = \frac{\overset{0}{\overset{\shortparallel}{j_-(\frac{1}{2})}} - j_-(\rho_+)}{\frac{1}{2} - \rho_+} < 0, \quad s(0) = 0$$

$$\Rightarrow \rho(x,t) = \begin{cases} \rho^+, & s(t) \le x \le 0 \\ \frac{1}{2}, & x \le s(t) \end{cases}$$



$c_+(0) = 1$
rarefaction
$\frac{x}{t} = c(\varphi) = 1 - 4\varphi$
$\varphi = \frac{1}{4}\left(1 - \frac{x}{t}\right)$