Norwegian University of Science and Technology
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TMA4195 Mathematical modelling Autumn 2012

Solutions to exam December 11, 2012

$\boxed{1}$ Dimensional matrix A:

	d	τ	μ	u	r
m	1	-1	-1	1	1
s	0	-2	-1	-1	0
kg	0	1	1	0	0

The d, μ, u columns are linearly independent, so d, μ, u are dimensional independent and core and possible core variables. By Buckingham's Pi-theorem there are

$$5 - \text{rank}\, A = 5 - 3 = 2$$

dimensional independent combinations, e.g.

$$\pi_1 = \frac{\tau}{d^a \mu^b u^c} = \ldots = \frac{\tau}{\mu}\frac{d}{u},$$
$$\pi_2 = \frac{r}{d}.$$

By Buckingham's Pi-theorem again, any dimensionally consistent relation

$$\tau = \Phi(d, \mu, u, r)$$

is equivalent to a relation

$$\Psi(\pi_1, \pi_2) = 0,$$

and solving for π_1 we find that $\pi_1 = C(\pi_2)$, or $\tau = C\left(\frac{d}{r}\right)\mu$.

$\boxed{2}$ We introduce the scales $c^* = Cc, x^* = Xx, t^* = Tt$. The natural scales are $C = M$ and $X = L$. The scaled equation is

$$\frac{M}{T}c_t = \frac{DM}{L^2}c_{xx} + \frac{UM}{L}c_x + rMc.$$

By assumption $c_t, c_{xx}, c_x, c, x, t \sim 1$, hence after dividing by M,

$$\frac{1}{T} \sim \frac{D}{L^2} + \frac{U}{L} + r = \frac{1}{L^2}\left(D + uL + rL^2\right).$$

Case (i) : $D \gg UL + rL^2$

The time scale is then given by $\frac{1}{T} \sim \frac{D}{L^2}$, and the scaled equation is

$$c_t = C_{xx} + \varepsilon_1 c_x + \varepsilon_2 c,$$

where $\varepsilon_1 = \frac{UT}{L} = \frac{UL}{D} \ll 1$ and $\varepsilon_2 = rT = \frac{rL^2}{D} \ll 1$.

Case (ii) : $rL^2 \gg D + UL$

The time scale is given by $\frac{1}{T} \sim r$, and the scaled equation is

$$c_t = \varepsilon_1 c_{xx} + \varepsilon_2 c_x + c,$$

where $\varepsilon_1 = \frac{Dr}{L^2} \ll 1$ and $\varepsilon_2 = \frac{Ur}{L} \ll 1$.

$\boxed{3}$ (i) could model two competing populations of size n_1 and n_2. When $n_2 = 0$, the n_1-equation is the logistic equation with bounded population, while if $n_1 = 0$, the n_2-equation gives exponential growth.

(ii) could model two populations that are mutually beneficial, e.g. as in symbiosis.

We find equilibrium points given by

$$\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} n_1(1 - n_1 - an_2) \\ cn_2(1 - n_1) \end{pmatrix} = 0,$$

that is, $(n_1, n_2) \in \{(0,0), (1,0)\}$. We examine the stability of the equilibrium points. To that end consider the Jacobi matrix of $\mathbf{F}$,

$$D\mathbf{F}(n_1, n_2) = \begin{bmatrix} 1 - 2n_1 - an_2 & -an_1 \\ -cn_2 & c(1 - n_1) \end{bmatrix}.$$

By inserting the first equilibrium point we get

$$D\mathbf{F}(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix},$$

with at least one positive eigenvalue ($\lambda = 1$). This implies that $(0,0)$ is unstable. The second equilibrium point gives

$$D\mathbf{F}(1,0) = \begin{bmatrix} -1 & -a \\ 0 & 0 \end{bmatrix},$$

where $\max\{\mathrm{Re}(\lambda)\} = 0$ and no conclusion.

$\boxed{4}$ This is a singular perturbation problem. The only boundary point is $t = 0$, and the boundary layer is there. We solve the problem as follows.

(i) Find the outer solution.

(ii) Find the boundary layer thickness and the rescaled equations.

(iii) Find the inner solution.

(iv) Mathcing in the intermediate region.

(v) The uniform approximation.

Implementing the above strategy.

(i) The outer solution solves the equations with $\varepsilon = 0$,

$$\begin{aligned} \dot{x}_O &= -x_O + (x_O + 1)(y_O - 1), \\ 0 &= x_O - (x_O + 1)y_O. \end{aligned}$$

The solution is

$$x_O(t) = Ce^{-t} - (1 - e^{-t}),$$
$$y_O(t) = \frac{x_O(t)}{x_O(t) + 1},$$

where we have used the technique of integrating factor to determine $x_O(t)$.

(ii) We rescale the equation $t = \delta\tau$, $X(\tau) = x(t)$ and $Y(\tau) = y(t)$. Inserting into the equations,

$$\frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}\tau}X = -X + (X+1)(Y-1),$$
$$\frac{\varepsilon}{\delta}\frac{\mathrm{d}}{\mathrm{d}\tau}Y = X - (X+1)Y.$$

If we balance terms in the first equation we get $\delta = 1$ and recover the outer solution. If we, on the other hand, balance terms in the second equation we get $\delta = \varepsilon$. Hence $\delta = \varepsilon$ is the other time scale.

(iii) The rescaled equations are

$$\dot{X} = \varepsilon\left(-X + (X+1)(Y-1)\right), \qquad X(0) = 1$$
$$\dot{Y} = X - (X+1)Y, \qquad Y(0) = 0.$$

We get the inner solution be letting $\varepsilon = 0$, hence

$$X_I(t) = 1,$$
$$Y_I(t) = \frac{1}{2}\left(1 - e^{-2t}\right).$$

(iv) The matching condition is

$$\lim_{t\to 0^+}\big(x_O(t), y_O(t)\big) = \lim_{\tau\to\infty}\big(X_I(\tau), Y_I(\tau)\big).$$

We calculate the limits using the solutions and get that $C = 1$.

(v) The uniform approximation is calculated by summing the inner and outer solution and subtracting the intermediate value. Thus

$$x_u(t) = 2e^{-t} - 1,$$
$$y_u(t) = 1 - \frac{1}{2}e^{t} - \frac{1}{2}e^{-2\frac{t}{\varepsilon}}.$$

$\boxed{5}$ The kinematic speed is

$$c(\rho) = j'(\rho) = 1 - 2\rho,$$

and the characteristics are given by

$$\dot{x} = c(z), \qquad x(t_0) = x_0,$$
$$\dot{z} = 0, \qquad z(t_0) = \rho(x_0, t_0).$$

The solution of this system is

$$z(t) = \rho(x_0, t_0),$$
$$x(t) = x_0 + (t - t_0)c\big(\rho(x_0, t_0)\big).$$

In $x \geq 0$, inflow boundary corresponds to $c \geq 0$. To find a boundary condition at $x = 0$, we convert to Dirichlet condition

$$\frac{3}{16} = j(\rho) = \rho(1-\rho),$$

which gives $\rho = \frac{1}{4}$ or $\rho = \frac{3}{4}$. Since $c(\frac{1}{4}) > 0$ and $c(\frac{3}{4}) < 0$, and we can only impose boundary conditions at inflow, we take

$$\rho = \frac{1}{4} \text{ at } x = 0, t > 0.$$

The characteristics are

$$x(t) = \begin{cases} x_0 + tc\left(\frac{1}{8}\right), & t_0 = 0, x_0 \geq 0, t > 0, \\ (t-t_0)c\left(\frac{1}{4}\right), & x_0 = 0, t > t_0 > 0, \end{cases}$$

$$z(t) = \begin{cases} \frac{1}{8}, & t_0 = 0, x_0 \geq 0, t > 0, \\ \frac{1}{4}, & x_0 = 0, t > t_0 > 0. \end{cases}$$

In the dead sector there is a rarefaction wave solution.

6 a) Conservation of mass in R

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R \rho \,\mathrm{d}V = \int_{\partial R} \rho \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma. \tag{1}$$

Observe that

$$\int_R \rho \,\mathrm{d}V = \rho \int_0^1 \mathrm{d}y \int_{\bar{z}}^{\bar{z}+\Delta z} \int_0^{d(z,\bar{t})} \mathrm{d}x\mathrm{d}z,$$

and since d is smooth,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R \rho \,\mathrm{d}V = \rho \int_{\bar{z}}^{\bar{z}+\Delta z} d_t(z,\bar{t})\,\mathrm{d}z.$$

As $\mathbf{v}\cdot\mathbf{n} = -u\mathbf{e}_z\cdot\mathbf{n}$,

$$\begin{aligned}\int_{\partial R} \rho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma &= \int_{\partial R\cap\{z=\bar{z}\}} \rho(-u)(-1)\,\mathrm{d}\sigma \\ &\quad + \int_{\partial R\cap\{z=\bar{z}+\Delta z\}} \rho(-u)(-1)\,\mathrm{d}\sigma \\ &= \rho\big((ud)(\bar{z}) - (ud)(\bar{z}+\Delta z)\big).\end{aligned}$$

If we divide (1) by $p\Delta z$, we find

$$\frac{1}{\Delta z}\rho\int_{\bar{z}}^{\bar{z}+\Delta z} d_t\,\mathrm{d}z = \frac{\rho\big((ud)(\bar{z}) - (ud)(\bar{z}+\Delta z)\big)}{\Delta z}.$$

Let $\Delta z \to 0$, use definition of derivative and the fundamental theorem of calculus to see that

$$\frac{\partial}{\partial t}d = \frac{\partial}{\partial z}(ud).$$

b) Fix $\bar{t} > 0$ and $\bar{z} \in [0, 1)$, and let $\Delta z > 0$ be such that $\bar{z} + \Delta z < 1$. If $d(\bar{z}, \bar{t}) = 0$ then

$$\int_R -\rho g \mathbf{e}_z \, \mathrm{d}V = -\int_{R \cap \{x=0\}} C \frac{\mu u}{d} \mathbf{e}_z \, \mathrm{d}\sigma.$$

This implies that

$$\rho g d(\bar{z}, \bar{t}) = C\mu \left(\frac{u}{d} \right) (\bar{z}, \bar{t}),$$

and hence

$$u = \frac{\rho g}{C\mu} d^2 = K d^2.$$

By a), d then satisfies

$$d_t - \left(K d^3 \right)_z = 0.$$

Method of characteristics $(z(t) = d\big(x(t), t\big))$

$$\begin{aligned} &\dot{x} = -2Kz^2 = c(z), \qquad x(0) = x_0, \\ &\dot{z} = 0, \qquad z(0) = d(x_0, 0). \end{aligned}$$

The characteristics are then

$$\begin{aligned} x &= x_0 + tc\big(d(x_0, 0)\big), \\ z &= d(x_0, 0). \end{aligned}$$

Since $z_1 < z_2$ implies that $d(z_1, 0) < d(z_2, 0)$, which again implies that $c\big(d(z_1, 0)\big) > \big(d(z_2, 0)\big)$, the characteristics will develop shock. The shock speed via the Rankine-Hugoniot condition is

$$\dot{s} = -2K \frac{\rho^2(z_1, 0) - \rho^2(z_2, 0)}{\rho(z_1, 0) - \rho(z_2, 0)} < 0$$

so the shock moves downwards.