Dimensional Analysis - Concepts

- **Physical quantities:** $R_j = v(R_j)[R_j] =$ value $\cdot$ unit, $\quad j = 1, \ldots, m$.
- **Units:** $[R_j] = F_1^{a_{1j}} \cdots F_n^{a_{nj}}, \quad F_1, \ldots, F_n$ fundamental units.
- **Dimension matrix** of $R_1, \cdots, R_m$: $A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$
- **Change of units** $\Rightarrow$ change of values:

  Lemma 1: $F_i = x_i \hat{F}_i$, $x_i > 0 \quad \Rightarrow \quad \hat{v}(R_j) = x_1^{a_{1j}} \ldots x_n^{a_{nj}} v(R_j)$
- **Dimensionless combination:** $\pi = R_1^{\lambda_1} \cdots R_m^{\lambda_m}$ if $\vec{\lambda} \neq 0$, $[\pi] = 1$
- **Dimensionally independent** $R_1, \ldots, R_s$ if no dimensionless comb'ns exist
- **Physical relations** $\Phi(R_1, \ldots, R_m) = 0$ are dimensionally consistent, i.e.

$$\Phi(v(R_1), \ldots, v(R_m)) = 0 \quad \Leftrightarrow \quad \Phi(\hat{v}(R_1), \ldots, \hat{v}(R_m)) = 0$$

for all changes of units $\hat{F}_i$. (consistent under change of units)

Dimensional Analysis - Buckingham's pi-theorem

(A1) $F_1, \ldots, F_n$ are fundamental units
(A2) $R_1, \ldots, R_m$ are physical quantities
(A3) $\Phi(R_1, \ldots, R_m) = 0$ is dimensionally consistent.

Lemma 2: Let $r = \operatorname{rank} A$, then $R_1, \ldots, R_m$ have $m - r$ independent dimensionless combinations.

OBS: The rank = number of linearly independent collumns in the matrix.

Buckingham's pi-theorem:

If (A1) – (A3) hold, then there are $m - r$ independent dimensionless combinations, and for any set of $m - r$ independent dimensionless combinations $\pi_1, \ldots, \pi_{m-r}$, there is a relation Ψ such that

$$\Phi(R_1, \ldots, R_m) = 0 \quad \Leftrightarrow \quad \Psi(\pi_1, \ldots, \pi_{m-r}) = 0,$$

where $r = \operatorname{rank} A$ and A is the $n \times m$ dimension matrix of $R_1, \ldots, R_m$.

It remains to prove Lemma 2 and the Pi-theorem.

Scaling and non-dimensionalizing

Produce dimensionless $O(1)$ variabels and dim.less eq'ns with terms $\lesssim 1$

Scaling a variable u^*: $\quad u^* = Uu \quad$ where

scaled variable: $u \sim 1, \qquad [u] = 1$

scaling constant: $U \sim \max|u^*|, \quad [U] = [u^*]$

Scaling/nondimensionalizing an equation:

1. scaling all variables in the equation
2. dividing the resulting equation by $\sim$ biggest coefficient.

Finding scales:

- look for combinations of the parameters
- balance 2 dominating ("biggest") terms in the equation (using that all scaled variables should be O(1))
- solve a reduced problem to find estimates
- typical time scale for $u^*(t^*)$: $\quad T = \dfrac{\max|u^*|}{\max\left|\frac{du^*}{dt^*}\right|}$

Remarks on scaling

Remark 1: 2 dominating terms balanced

$\Rightarrow$ their coefficients are equal and $\sim$ biggest in equation.

$\Rightarrow$ dividing scaled equation by this coefficient:

All variables and coefficients become

dimensionless,

$\lesssim 1$, and

2 coefficients $= 1$.

Remark 2: Different situations $\Rightarrow$ different scales

$\max|x^*|$, $\max|t^*|$, $\max|u^*|$, etc., and the dominating terms in the equation depend on the situation.

Remark 3: Advantages of scaling:

- minimize the number of parameters/coefficients (experiments!!),
- normalize all variables and coefficients,
- reduce round-off errors in subsequent numerical calculations,
- make small terms visible $\Rightarrow$ easy to do approximations/perturbation.

Regular Perturbation

Given scaled(!) equation: $\ddot{x} = -\frac{1}{(1+\varepsilon x)^2}, \qquad 0 < \varepsilon \ll 1.$

1. Perturbation Assumption: $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$
2. Insert into equation, expand as power series in ε:

$$\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots = -\frac{1}{\Big(1 + \varepsilon(x_0 + \varepsilon x_1 + \dots)\Big)^2}$$

$$= -1 + 2\varepsilon\Big(x_0 + \varepsilon x_1 + \dots\Big) - 3\varepsilon^2\Big(x_0 + \varepsilon x_1 + \dots\Big)^2 + \dots$$

$$= -1 + \varepsilon 2x_0 + \varepsilon^2(2x_1 - 3x_0^2) + \dots$$

3. Equate terms of same order in $\varepsilon \longrightarrow$ equations for $x_0, x_1, \dots$:

$$\begin{aligned} O(1): &\quad \ddot{x}_0 = -1 \\ O(\varepsilon): &\quad \ddot{x}_1 = 2x_0 \\ O(\varepsilon^2): &\quad \ddot{x}_2 = 2x_1 - 3x_0^2 \end{aligned}$$

5. Solve these equations recursively for $x_0, x_1, x_2, \dots$.

Singular Perturbation

Signs:

- Multiple time/space scales
- Initial/boundary layers
- Small paramater multiplying principal term
- Naive approximation changes problem completely

Facts:

- No single scale is good for complete resolution of problem
- Different regions, different scales, different (re)scaled equations
- Scales found by balancing terms in equation
- In each region regular perturbation works
- Matching conditions between perturbation sol'ns of different regions

Singular Perturbation – first approximation

$$\varepsilon y'' + 2y' + y = 0,\ 0 < x < 1; \quad y(0) = 0,\ y(1) = 1; \quad 0 < \varepsilon \ll 1.$$

1. Guess where boundary layer is: $x = a$. Here $a = 0$.
2. Outer solution y_O. Set $\varepsilon = 0$ and solve equation and boundary condition outside boundary layer:
$$2y_O' + y_O = 0;\ y_O(1) = 1 \implies y_O(x) = e^{\frac{1}{2} - \frac{x}{2}}.$$
3. Find length of boundary layer δ (the other *consistent* space scale) by balancing terms $\to \dots\ \delta = \varepsilon$.
4. Rescale equation: $(x, y) = (\delta\xi, Y) \to Y''(\xi) + 2Y'(\xi) + \varepsilon Y(\xi) = 0$
5. Inner solution y_I. Set $\varepsilon = 0$ and solve rescaled equation and boundary condition inside boundary layer.
$$y_I'' + 2y_I = 0,\ y_I(0) = 0 \implies y_I(\xi) = C(1 - e^{-2\xi}).$$
6. Matching. $y_O \approx y_I$ in intermediate region $\underset{\varepsilon \to 0}{\to} \dots C = e^{\frac{1}{2}}$ (approx'n!)
7. Uniform solution: $y_U(x) = y_O(x) + y_I(\frac{x}{\delta}) - \lim\limits_{\varepsilon \to 0} y_O(\Theta\eta)$

Equilibrium points

1. **Equilibrium point** = constant solution u_e (e.g. of ODEs or PDEs)

2. An equilibrium point u_e is **stable** if all solutions starting near u_e, remain near u_e for all $t \geq 0$.

3. **Linear stability analysis**

   1. Write solution $u = u_e + \tilde{u}$, $\tilde{u}$ small perturbation

   2. Linearize equation(s) about u_e:

      insert $u = u_e + \tilde{u}$ into equation(s)

      drop small(=non-linear in $\tilde{u}$) terms

      Result: linear equation(s) for $\tilde{u}$,
      with equilibrium point $\tilde{u}_e = 0$.

   3. Check stability of $\tilde{u}_e = 0$ (linearized equation(s)!!)

   4. Conclusion: $\tilde{u}_e = 0$ stable/unstable indicate that u_e stable/unstable.

4. Over time all physical systems tend to be at their stable equilibrium solutions! (... always small disturbances ...)

Aggregation of Amoeba

Background: *Lack of food → amoeba produce attractant and aggregate.*

Question:

Can onset of aggregation be caused by simple, uninteligent mechanism?

Model near onset of aggregation:

- Physical quantities:
  $a(x,t)$, $c(x,t)$ = amoeba, attractant densities; parameters
- Modelling (conservation+diffusion+attraction+production+decay):

$$(1) \qquad a_t = \frac{\partial}{\partial x}\Big(ka_x - lac_x\Big), \qquad c_t = Dc_{xx} + q_1 a - q_2 c.$$

- Equilibrium points (=constant solutions):
  Constants (a_0, c_0) such that $q_1 a_0 = q_2 c_0$.
- Linearize equation around (a_0, c_0):
  $a = a_0 + \tilde{a}$, $c = c_0 + \tilde{c}$; $\tilde{a}, \tilde{c}$ small; drop small terms

$$(2) \qquad \tilde{a}_t = \frac{\partial}{\partial x}\Big(k\tilde{a}_x - la_0\tilde{c}_x\Big), \qquad \tilde{c}_t = D\tilde{c}_{xx} + q_1\tilde{a} - q_2\tilde{c}.$$

Aggregation of Amoeba

$$(2) \qquad \tilde{a}_t = \frac{\partial}{\partial x}\Big(k\tilde{a}_x - la_0\tilde{c}_x\Big), \qquad \tilde{c}_t = D\tilde{c}_{xx} + q_1\tilde{a} - q_2\tilde{c}.$$

- Particular solutions of (2): Fourier modes/eigenfunctions

$$(\tilde{a}, \tilde{c}) = e^{\alpha t}\cos(\beta x)(C_1, C_2)$$

solve (2) iff

1. $\alpha^2 + b\alpha + c = 0$ for

$$b = k\beta^2 + D\beta^2 + q_2 \quad \text{and} \quad c = kq_2\beta^2 + kD\beta^4 - q_1 la_0\beta^2,$$

2. (C_1, C_2) satisfy two linear equations (last time).

Every $\beta \in \mathbb{R} \to$ two real α + linear subspace of solutions (C_1, C_2)

- Boundedness of solutions of (2):

$$(\tilde{a}, \tilde{c}) \text{ bounded } \Leftrightarrow \alpha \le 0 \Leftrightarrow c \ge 0 \Leftrightarrow \boxed{kD\beta^2 + kq_2 \ge q_1 la_0}$$

- Unboundedness: $kq_2 < q_1 la_0 \quad \Rightarrow \quad (\tilde{a}, \tilde{c})$ blow up at ∞ for $\beta \ll 1$
  - Take C_1, C_2 arbitrarily small $\Rightarrow (0,0)$ unstable equilibrium pt. of (2)
  - Indicate that (a_0, c_0) ($\Leftrightarrow (\tilde{a}, \tilde{c}) = (0,0)$) unstable equi. pt. of (1)

Convervation in Continuum Mechanics

The transport theorem: $R(t) = \{x(t) : \dot{x} = \vec{v}(x,t), x(t_0) = x_0 \in R(t_0)\}$

$$\frac{d}{dt}\int_{R(t)} f(x,t)\,dx\Big|_{t=t_0} = \frac{d}{dt}\int_{R(t_0)} f(x,t)\,dx\Big|_{t=t_0} + \int_{\partial R(t_0)} f(x,t_0)\,(\vec{v}\cdot\vec{n})\,d\sigma$$

Conservation of mass and momentum:

(1) $$\frac{d}{dt}\int_R \rho\,dx + \int_{\partial R} \rho\,(\vec{v}\cdot\vec{n})\,d\sigma = \int_R q\,dx.$$

(2) $$\frac{d}{dt}\int_R \rho\vec{v}\,dx + \int_{\partial R} \rho\vec{v}\,(\vec{v}\cdot\vec{n})\,d\sigma \overset{\text{Transp.}}{\underset{\text{thm.}}{=}} \frac{d}{dt}\int_{R(t)} \rho\vec{v}\,dx \overset{\text{Newton's}}{\underset{\text{2nd law}}{=}} \underbrace{\int_{R(t)} \vec{f}_B\,dx}_{\text{body forces}} + \underbrace{\int_{\partial R(t)} \vec{f}_S\,d\sigma}_{\text{surface forces}}$$

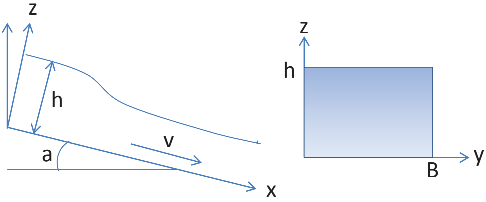
Newtonian fluid: $\vec{f}_S = T\cdot\vec{n},\ T_{ij} = -(p + \frac{2}{3}\mu\nabla\cdot\vec{v})\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})$

Differential form - the Navier-Stokes equations:

(1') $\rho_t + \nabla\cdot(\rho\vec{v}) = q$

(2') $(\rho v_i)_t + \nabla\cdot(\rho v_i\vec{v}) = f_{B,i} - \frac{\partial p}{\partial x_i} + \mu\big(\nabla^2 v_i + \frac{1}{3}\frac{\partial}{\partial x_i}(\nabla\cdot\vec{v})\big),\ i = 1,2,3$

Flow in rivers – the shallow water equations



Assumptions:

1. v, h depend only on x, t; a small; $\rho = const.$
2. Dominant forces in x-direction:
   1. Gravity $\vec{f}_g \cdot \vec{e}_x = \rho g \sin a$
   2. Hydrostatic pressure $\vec{f}_p \cdot \vec{e}_x \approx -\rho g(h - z)(\vec{n} \cdot \vec{e}_x)$
   3. Bottom friction $\vec{f}_f \cdot \vec{e}_x = -\rho C_f v^2$

Control volume: $R = \{(x, y, z) : x \in [x_0, x_0 + \Delta x], y \in [0, B], z \in [0, h(x, t_0)]\}$

Conservation of mass and momentum in x-direction:

$$\frac{d}{dt} \int_R \rho \, dx + \int_{\partial R} \rho \, (\vec{v} \cdot \vec{n}) \, d\sigma = 0,$$

$$\frac{d}{dt} \int_R \rho v \, dx + \int_{\partial R} \rho v \, (\vec{v} \cdot \vec{n}) \, d\sigma = \int_R \vec{f}_g \cdot \vec{e}_x \, dx + \int_{\partial R} (\vec{f}_p + \vec{f}_f) \cdot \vec{e}_x \, d\sigma.$$

Flow in rivers – the shallow water equations

Compute all integrals, divide by common factor ρB:

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} h\,dx + [(vh)(x_0+\Delta x) - (vh)(x_0)] = 0,$$

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} vh\,dx + [(v^2h)(x_0+\Delta x) - (v^2h)(x_0)]$$
$$= \int_{x_0}^{x_0+\Delta x} gh\sin a\,dx - \frac{g}{2}[h^2(x_0+\Delta x) - h^2(x_0)] - \int_{x_0}^{x_0+\Delta x} C_f v^2\,dx.$$

Divide by Δx, let $\Delta x \to 0$:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0,$$

$$\frac{\partial}{\partial t}(vh) + \frac{\partial}{\partial x}\left(v^2h + \frac{g}{2}h^2\right) = gh\sin a - C_f v^2.$$

This is the shallow water equations or St. Venant system.