TMA 4195
Mathematical Modeling

Supplementary Notes, Part 1:

Dimensional Analysis, Scaling, and Regular Perturbation

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Contents

1 DIMENSIONAL ANALYSIS

1.1 The Basis of Dimensional Analysis

Dimensional Analysis is a technique based on two simple axioms about nature:

- *All relations between physical quantities must be dimensionally correct*
- *No physical relation should depend on any particular set of units*

Even if these axioms sound trivial and obvious, they lead to a powerful, simple and quite useful tool in mathematical modelling. At the same time, it illustrates how important it is to uncover the mathematical essence in two general and apparently vague statements. Let us now investigate the axioms in somewhat more detail. A physical quantity has

- *dimension*
- *unit*
- *numerical value*

Dimension means, e.g. *Length*, *Time*, *Mass*, or combinations. The dimension of a physical quantity is given once and for all and will never change. When we work with physical quantities we need units, and as we know, there is a huge amount of units. For a length we have cm, m, km, foot, inch etc. When the unit changes, the numerical value of the quantity also changes, as illustrated in Fig. 1.

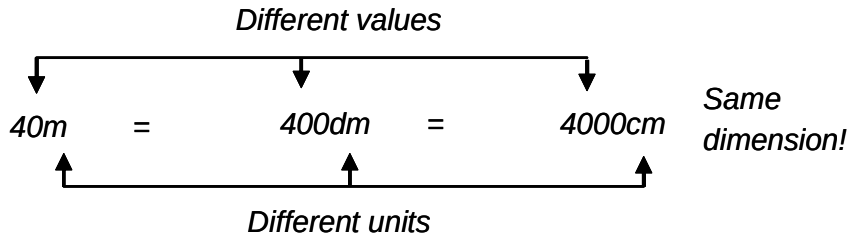

Figure 1: The numerical value changes when the unit changes, but the physical dimension remains the same.

Let R be a physical quantity. The reader should have observed by now that we have not really defined what a physical quantity is. *Wikipedia* defines a physical quantity as a physical property that can be quantified in terms of numbers. Thus, the mass of Earth is a physical quantity whereas a dice showing a 6 for a rock concert is not. It is convenient to have a notation for the unit of R, and we shall write this as $[R]$. The value of R when we use a certain unit is denoted $v(R)$. This is not a standard notation, but convenient for the moment. Hence, R has a unit and a numerical value,

$$R = v\,(R)\,[R]\,.$$

Quantity	Dimension symbol	SI unit
Mass	M	kilo (kg)
Length	L	meter (m)
Time	T	second (s)
Electric current	I	ampere (A)
Absolute temperature	Θ	Kelvin (K)
amount of substance	N	Mole (mol)
luminous intensity	J	Candela (Cd)

Table 1: Fundamental physical units in the SI-system. Exact definitions may be found in Wikipedia.

That a physical relation or equation is *dimensionally correct* means we do not add apples and pears, or that each side of an equality do not have different units. It is obvious that the well-known formula $S = \frac{1}{2}gt^2$ is dimensionally correct since

$$\begin{aligned} [S] &= \mathrm{m}, \\ [g] &= \mathrm{m/s^2}, \\ [t] &= \mathrm{s}. \end{aligned} \tag{1}$$

On the contrary, you often find, even in textbooks, equations of the form

$$S = 4.9 \times t^2. \tag{2}$$

In order to be meaningful t must be measured in seconds and s in metres, and it will not work for other pairs of units. Such relations should not be used. If you can not state your equations in a dimensionally correct form, you have probably not a clear idea of what is going on. In physics there is a set of dimensions forming so-to-speak the *atoms*. None of these depend on the others and the dimensions along with corresponding units in the *International System of Units* (abbreviated the *SI-system*) are listed in Table 1. All physical quantities have units which are power combinations of the basic SI-system units (this may be considered as the definition of a physical quantity).

1.2 Buckingham's Pi-theorem

Buckingham's Pi-theorem extracts the mathematical content of the two axioms in the introduction. The somewhat strange name comes from the dimensionless variables that we end up with when we apply the theorem. These are often referred to as π_1, π_2, $\cdots$ (Π is also used instead of π).

Let us look at what *a relationship between physical quantities* means. A relationship is a relation or a formula, that is, an equation that we can write

$$\Phi(R_1, R_2, \cdots, R_M) = 0, \tag{3}$$

where Φ is a certain function. We could also write this relation in other ways, *e.g.*

$$R_1 = \Psi(R_2, \cdots, R_M). \tag{4}$$

If we choose a set of fundamental units, and uses these in a consistent way for all the involved variables, we should now have

$$\Phi(v(R_1), v(R_2), \cdots, v(R_M)) = 0.$$

In practice we could, for example, specify $v(R_2), \cdots, v(R_M)$, and then calculate $v(R_1)$ from the relationship. However, it may well happen that there is no valid relationship between $v(R1), \cdots, v(R_M)$. If we consider S and t where $[S] = \mathrm{m}$ and $[t] = \mathrm{s}$, there is no valid physical relation that contains *only* these two variables. We need at least one more quantity, as in the well-known relations $S = gt^2/2$ or $S = Vt$. By simply looking at the quantities and their units, it is possible to decide whether they at all can be combined into a sensible relation.

To investigate this further, it is smart to create a so-called *dimension matrix* containing the exponents of the fundamental units in the units for the quantities we have. If $[S] = \mathrm{m}$, $[t] = \mathrm{s}$ and $[g] = \mathrm{ms}^2$, the dimension matrix will be

	S	t	g
m	1	0	1
s	0	1	-2

. (5)

Let us already here point out that we use the familiar units in the first column. Actually, it would be more correct to use universal dimension assignments such as L for *length*, T for *time*, M for *mass*, etc. This is used in many textbooks. The dimension matrix surveys what the dimensions of involved variables are.

Let now, in general, $F_1, F_2, \cdots, F_N$ denote the fundamental units in Table 1. The units of any physical quantity may be expressed by means of these, for example, the unit for *energy* is "$\mathrm{kgm}^2/\mathrm{s}^2$". Generally, we may thus write

$$\begin{aligned} [R_1] &= F_1^{a_{11}} F_2^{a_{21}} \cdots F_N^{a_{N1}}, \\ &\vdots \\ [R_M] &= F_1^{a_{1M}} F_2^{a_{2M}} \cdots F_N^{a_{NM}}. \end{aligned} \tag{6}$$

This gives the dimension matrix $\mathbf{A}$:

	R_1	R_2		$\cdots$	R_M
F_1	a_{11}			$\cdots$	a_{1M}
F_2	$\vdots$		$\mathbf{A}$		$\vdots$
$\vdots$				$\ddots$	
F_N	a_{N1}		$\cdots$		a_{NM}

. (7)

We say that $R_1, \cdots, R_r$ have *independent dimension* if it is impossible to make a (non-trivial) dimensionless combination of $R_1, \cdots, R_r$ of the form

$$R_1^{\lambda_1} R_2^{\lambda_2} \times \cdots \times R_r^{\lambda_r}. \tag{8}$$

In order to see what this means, we determine the unit of this expression:

$$\begin{aligned} \left[R_1^{\lambda_1} R_2^{\lambda_2} \times \cdots \times R_r^{\lambda_r}\right] = F_1^{a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots a_{1r}\lambda_r} \times \\ \times F_2^{a_{21}\lambda_1 + a_{22}\lambda_2 + \cdots a_{2r}\lambda_r} \times \cdots \\ \cdots \times F_N^{a_{N1}\lambda_1 + a_{N2}\lambda_2 + \cdots a_{Nr}\lambda_r}. \end{aligned} \tag{9}$$

That $R_1, \cdots, R_r$ have independent dimensions means that the equation system

$$\begin{array}{c} a_{11}\lambda_1 + a_{12}\lambda_2 + \cdots a_{1r}\lambda_1 = 0, \\ a_{21}\lambda_1 + a_{22}\lambda_2 + \cdots a_{2r}\lambda_r = 0, \\ \vdots \\ a_{N1}\lambda_1 + a_{N2}\lambda_2 + \cdots a_{Nr}\lambda_r = 0, \end{array} \tag{10}$$

only has the trivial solution

$$\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0. \tag{11}$$

We may recall from the theory of linear equations that this will happen if and only if the matrix columns,

$$\left[\begin{array}{c} a_{11} \\ \vdots \\ a_{N1} \end{array}\right], \left[\begin{array}{c} a_{12} \\ \vdots \\ a_{N2} \end{array}\right], \cdots, \left[\begin{array}{c} a_{1r} \\ \vdots \\ a_{Nr} \end{array}\right], \tag{12}$$

are *linearly independent*. Thus, we have proved that $R_1, \cdots, R_r$ have independent dimensions if and only if the corresponding dimensional matrix has linearly independent columns. Since the vectors have unit N, we must have $r \leq N$ for this to be possible. Going back to the example above, we see that S and t have independent dimensions. The same applies for $\{S, g\}$ and $\{t, g\}$. On the contrary, S, t and g do not have independent dimensions.

If we now have a general dimension matrix, the maximum number of variables with independent dimension is equal to the maximum number of columns that are linearly independent. This is known in linear algebra as the *rank* of the matrix. Let us assume that we have organized ourselves so that $R_1, \cdots, R_r$ have independent dimension, and that r is the rank of A, rank(A). We may assume that $r < M$ (If $r = M$, all quantities have independent dimension and there will be no non-trivial physical relationship between them). From this assumption, we may use $R_1, \cdots, R_r$ to form combinations involving $R_{r+1}, \cdots, R_M$ such that

$$\begin{array}{c} \pi_1 = R_{r+1}/(R_1^\bullet \times \cdots \times R_r^\bullet), \\ \pi_2 = R_{r+2}/(R_1^\bullet \times \cdots \times R_r^\bullet), \\ \vdots \\ \pi_{M-r} = R_M/(R_1^\bullet \times \cdots \times R_r^\bullet), \end{array} \tag{13}$$

is dimensionless. Here "$^\bullet$" means suitable exponents so as to make the π-s dimensionless. If we then have a relation

$$\Phi(R_1, R_2, \cdots, R_M) = 0, \tag{14}$$

it is possible to replace $R_{r+1}, \cdots, R_M$ and arrange the expression such that we end with a new, but equivalent relation,

$$\Psi(R_1, R_2, \cdots, R_r, \pi_1, \cdots, \pi_{M-r}) = 0. \tag{15}$$

We now claim: *If $R_1, \cdots, R_r$ have independent dimensions, it is possible to choose a set of fundamental units such that the values $v(R_1), v(R_2), \cdots, v(R_r)$ become arbitrarily specified positive numbers*!

This is easy to see if we have just one quantity, *e.g.* $R_1 = 40$m. If we measure R_1 in centimeters, $v(R_1) = 4000$, while measured in kilometers,$v(R_1) = 0.04$, and so on.

If the claim is correct, we have obtained the following interesting situation. Whatever units we decide to use,

$$\Psi\left(v(R_1),v(R_2),\cdots,v(R_r),v(\pi_1),\cdots,v(\pi_{M-r})\right)=0. \tag{16}$$

While the first r variables may take any positive value depending on how we choose the units, the latest $M-r$ variables remain constant during this change. As a function of M variables Ψ will therefore always be completely unaffected by the values of the r first arguments. In other words, Ψ can really only depend on $\pi_1,\cdots,\pi_{M-r}$! We have thus reduced our relation with M variables, $\Phi(R_1,R_2,\cdots,R_M)=0$ to a new relation $\Psi(\pi_1,\cdots,\pi_{M-r})=0$ with only $M-r$ variables. Apart from the claim above, which is proved below, we have now proven

Buckingham's Pi-teorem:

If there exists a (physically proper) relation

$$\Phi(R_1,R_2,\cdots,R_M)=0 \tag{17}$$

between the quantities $R_1,R_2,\cdots,R_M$, there also exists an equivalent relation

$$\Psi(\pi_1,\cdots,\pi_{M-r})=0, \tag{18}$$

where r is the rank of the dimension matrix.

Note that the theorem *assumes* that there is a relationship between $R_1,R_2,\cdots$, and R_M. This has to be ensured, or at least assumed, before we apply the theorem. In fact, Buckingham's Pi-teorem may also prove that no such relation exists. Buckingham Pi-theorem reduces the number of parameters, and if none of the dimensionless π-s are completely redundant, $M-r$ is also the least possible number of variables we have in our problem.

In the proof above we applied $R_1,R_2,\cdots,R_r$ to create the dimensionless combinations. These variables are often called *core variables*. Usually there are several possibilities for the core variables, and what is appropriate depends on the problem.

It is easy to set up a formal procedure to determine the π-s (see Sec. 1.3.2 below). This is also found in the textbooks, *e.g.* [4], but most of the time it is just as easy to find the combinations by a simple inspection. Note that the number of dimensionless variables will be the same regardless the choice of core variables and combinations. In principle, all are equally valid.

In the remainder of this section, we shall, for those particularly interested, show the above claim.

We assume therefore that $R_1,\cdots,R_r$ has independent dimension, and that dimension matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{Nr} \end{bmatrix} \tag{19}$$

has rank r. A well-known proposition from linear algebra say that the *rank of a matrix* $\mathbf{A}$ *is the same as the rank of the transposed matrix* $\mathbf{A}^T$. It turns out that this sentence is exactly what we need to complete the proof.

Let us assume that, using the fundamental units $F_1,F_2,\cdots,F_N$, R_i has the values $v_F(R_i)$, $i=1,\cdots,r$. For another set of fundamental units, $G_1,G_2,\cdots,G_N$, the numerical values will be

$v_G(R_i),\ i = 1, \cdots, r$. Let $x_i = F_i/G_i$. Then,

$$\begin{aligned} R_i &= v_F(R_i) F_1^{a_{1i}} F_2^{a_{2i}} \cdots F_N^{a_{Ni}} \\ &= v_F(R_i) x_1^{a_{1i}} G_1^{a_{1i}} x_2^{a_{2i}} G_2^{a_{2i}} \cdots x_N^{a_{Ni}} G_N^{a_{Ni}} \\ &= v_F(R_i) x_1^{a_{1i}} x_2^{a_{2i}} \cdots x_N^{a_{Ni}} G_1^{a_{1i}} G_2^{a_{2i}} \cdots G_N^{a_{Ni}} \\ &= v_G(R_i) G_1^{a_{1i}} G_2^{a_{2i}} \cdots G_N^{a_{Ni}}. \end{aligned} \tag{20}$$

Thus,

$$v_F(R_i) x_1^{a_{1i}} x_2^{a_{2i}} \cdots x_N^{a_{Ni}} = v_G(R_i),\ i = 1, \cdots, r. \tag{21}$$

If we take the logarithm of both sides, we end up with a linear system of equations of the form

$$\begin{aligned} a_{11} \log(x_1) + \cdots + a_{N1} \log(x_N) &= \log\left(v_G(R_1)\right) - \log\left(v_F(R_1)\right), \\ a_{12} \log(x_1) + \cdots + a_{N2} \log(x_N) &= \log\left(v_G(R_2)\right) - \log\left(v_F(R_2)\right), \\ &\cdots \\ a_{1r} \log(x_1) + \cdots + a_{Nr} \log(x_N) &= \log\left(v_G(R_r)\right) - \log\left(v_F(R_r)\right). \end{aligned} \tag{22}$$

We recognize the coefficient matrix in this equation system as the transposed of the dimension matrix. Since we have r equations and, according to the proposition from linear algebra, r linearly independent columns in the coefficient matrix, this system will have solutions $\log(x_1), \cdots, \log(x_N)$ (not necessarily unique) *regardless* of the choice of the right-hand side. But this means precisely that we may first specify $v_G(R_1), \cdots, v_G(R_r)$ to whatever we want, then find $\log(x_1), \cdots, \log(x_N)$, then $x_1, \cdots, x_N$, and finally select custom G-units , $G_i = F_i/x_i$, $i = 1, \cdots, r$.

1.3 Some Applications of Dimensional Analysis

1.3.1 The First Atomic Bomb Explosion

The following example of the use of dimensional analysis has become a classic. The English physicist G. I. Taylor watched an amateur film of the first American atomic bomb explosion in the Nevada desert, and measured the radius (r) of the fireball as a function of time (t), see Fig. 2. He argued that r, apart from t, should depend on the energy (E) that is released in the explosion and the density (ρ) of the air, since the flame-front needs to accelerate the mass of the surrounding air. Thus, he assumed that we have a relation $\Phi(r, t, \rho, e) = 0$, and set up the following dimension matrix

	r	t	ρ	e
kg	0	0	1	1
m	1	0	-3	2
s	0	1	0	-2

(23)

Note that we can find the unit of energy, expressed in terms of the fundamental units, by consulting well-known formulas from physics. For example, we know that energy is force×distance, and that force is mass×acceleration. This gives us

$$[e] = \mathrm{Nm} = (\mathrm{kgm/s^2})\mathrm{m} = \mathrm{kgm^2/s^2}.$$

The dimension matrix above has rank 3, and, according to Buckingham there is $4 - 3 = 1$ dimensionless parameter. Since the equation is simply $\Psi(\pi_1) = 0$, we assume it has a unique solution such that we may write

$$\pi_1 = C, \tag{24}$$

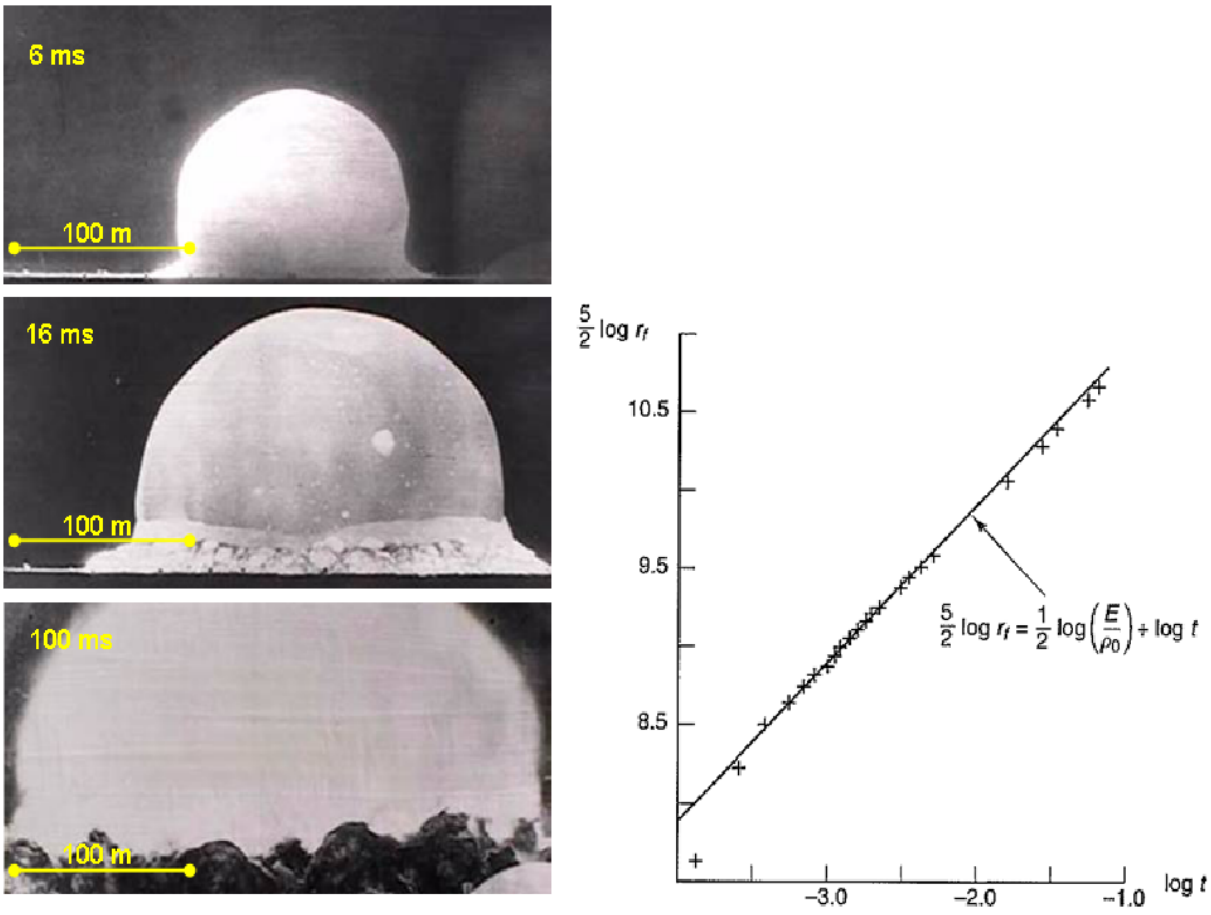

Figure 2: This series of images from the first atomic bomb explosjon shows a fireball growing with time. To the right is shown a copy of the figure G.I. Taylor published based on the whole sequence of images (See the book of Barenblatt, s. 47 – 50).

where C is an unknown constant. We may find π_1 by *trial and error*. First,

$$\left[\frac{e}{\rho}\right] = \frac{\mathrm{Nm}}{\mathrm{kg/m^3}} = \frac{\mathrm{kgm^2}}{\mathrm{s^2}}\frac{\mathrm{m^3}}{\mathrm{kg}} = \frac{\mathrm{m^5}}{\mathrm{s^2}}, \tag{25}$$

and then we observe that the following combination is dimensionless

$$\pi_1 = \frac{e}{\rho}\frac{t^2}{r^5}. \tag{26}$$

This gives us the following simple formula:

$$e = C\rho\frac{r^5}{t^2}. \tag{27}$$

We are not able to determine the constant C, but from an amateur film G.I. Taylor was able to find the ratio r^5/t^2, and by assuming $C = 1$ (best guess!), he got that the released energy $e \approx 10^{14}$J. It turned out that this was within a factor of 2 of the correct. The publication of the energy, which of course was "top secret", caused great confusion among the Americans when this was published as a letter in The Times.

1.3.2 A General Recipe for Finding Dimensionless Combinations

If it is difficult to see the dimensionless combinations directly, it is possible to put up a system of equations for the exponents. The method may be illustrated using the example above. Here we are looking for a dimensionless combination of the form $\pi = e^x\rho^y r^z t^u$ and must therefore determine $\{x, y, z, u\}$. Since we already know the units of the variables,

$$\begin{aligned}[\pi] &= [e^x\rho^y r^z t^u] \\ &= \left(\mathrm{kg}^x\mathrm{m}^{2x}\mathrm{s}^{-2x}\right)\left(\mathrm{kg}^y\mathrm{m}^{-3y}\right)\left(\mathrm{m}^z\right)\left(\mathrm{s}^u\right) \\ &= \mathrm{kg}^{x+y}\mathrm{m}^{2x-3y+z}\mathrm{s}^{-2x+u}.\end{aligned}$$

Here π should be dimensionless, and therefore,

$$\begin{aligned} x + y &= 0, \\ 2x - 3y + z &= 0, \\ -2x + u &= 0. \end{aligned}$$

There is no unique solution, but we choose $x = 1$, it follows easily that $y = -1$, $z = -5$, and $u = 2$, in other words, exactly what we already knew. There is no reason to use this cumbersome method if it is possible to see the result directly.

1.3.3 Pythagoras' Theorem

Since a right-angled triangle is completely determined by the length of the hypotenuse (c) and the smallest angle ($\alpha_{\min}$), there must be a relationship between surface area (A) of the triangle, the length of the hypotenuse and the angle,

$$\Phi(c, A, \alpha_{\min}) = 0. \tag{28}$$

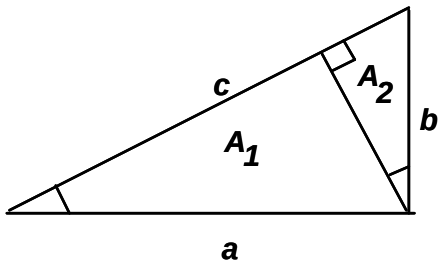

Figure 3: The area of the big triangle is the sum of the equally shaped smaller triangles

It is easy to set up the dimension matrix:

	A	c	$\alpha_{\min}$
m	2	1	0

.

(An angle is measured in radians, which is a ratio between two lengths and thus dimensionless). Since the rank of the matrix is 1, there are two dimensionless parameters, A/c^2 and $\alpha_{\min}$, and we end up with a relation $A = c^2 f(\alpha_{\min})$. From Fig. 3 it is obvious that $A = A_1 + A_2$ for the areas, and therefore $c^2 f(\alpha_{\min}) = a^2 f(\alpha_{\min}) + b^2 f(\alpha_{\min})$, or $c^2 = a^2 + b^2$, which is yet another proof of this famous result (apparently not yet listed among the 93 other proofs found at [6]).

1.3.4 Fluid Flows in Tubes

This example, giving an expression for the friction factor to use for fluids flowing in tubes, is quite famous and very useful in engineering. It requires some background in fluid mechanics.

We shall find an expression for the pressure drop in a cylindrical tube which contains a flowing fluid, and we assume that the variables in Table 2 are important.

Concerning *viscosity*, μ, this is a proportionality constant between *shear stress* (*force/area unit*), for example σ_{yx}, and changes in the speed per. unit length, $\partial u/\partial y$, normal to the force direction (consult a textbook in fluid mechanics or *Internet* for more information). For a so-called *Newtonian fluid* (like water and air), $\sigma_{yx} = \mu \partial u/\partial y$. Thus, the unit for μ is:

$$[\mu] = \frac{[\sigma_{yx}]}{[\partial u/\partial y]} = \frac{(\mathrm{kgm/s^2})/\mathrm{m^2}}{(\mathrm{m/s})/\mathrm{m}} = \frac{\mathrm{kg}}{\mathrm{ms}}. \tag{29}$$

The measure of the wall roughness (e) could, *e.g.* be the typical standard deviation around the mean (think of a cement tube with rough walls). Clearly, the size of e may vary over several orders of magnitude from very smooth glass tubes, to steel pipes, cement tubes or hydropower tunnels in rocks! From Table 2 we may derive the dimension matrix in Table 3.

We see that the rank is 3 (the maximum it can be), and consequently, we have $7 - 3 = 4$ dimensionless quantities. In this example it is not very smart to use P as a core variable because we want to express P in terms of the other variables. One possible choice for core variables is $\{V, D, \rho\}$ since

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & -3 \\ -1 & 0 & 0 \end{vmatrix} \neq 0, \tag{30}$$

Quantity	Name	Unit
Pressure	P	$\mathrm{N/m^2}$=$\mathrm{kg/s^2m}$
Mean fluid velocity	V	m/s
Tube diameter	D	m
Tube length	L	m
wall roughness	e	m
Viscosity	μ	kg/ms
Density of the fluid	ρ	$\mathrm{kg/m^3}$

Table 2: Quantities that may be included in the expression for the pressure drop in the pipe.

	P	V	D	L	e	μ	ρ
kg	1	0	0	0	0	1	1
m	-1	1	1	1	1	-1	-3
s	-2	-1	0	0	0	-1	0

Table 3: Dimension matrix for the variables in the expression for the pressure loss.

and the columns are thus linearly independent.

The next step is to form dimensionless combinations where the remaining variables are included. It is easy to check that the following combinations are possible choices:

$$\begin{aligned} \pi_1 &= P/(v^2\rho), \\ \pi_2 &= L/D, \\ \pi_3 &= e/D, \\ \pi_4 &= vD\rho/\mu. \end{aligned} \tag{31}$$

(There are other possibilities, but since we are not the first ones to carry out this exercise, we show only the most useful one). Since we want P to be expressed by the other variables, it is reasonable to think of a relationshipof the form

$$\pi_1 = \Phi(\pi_2, \pi_3, \pi_4). \tag{32}$$

Now it is also reasonable to assume (and this is verified by experiments) that the pressure drop is proportional to the tube L. Hence, it should be possible to write

$$\pi_1 = \pi_2 \Phi_2(\pi_3, \pi_4), \tag{33}$$

or

$$P = \frac{L\rho V^2}{D}\Phi_2(\frac{e}{D}, \frac{VD\rho}{\mu}). \tag{34}$$

In fluid mechanics it is common to replace Φ_2 with $2f_F$, where f_F is called *Fanning's friction factor,*

$$f_F = f_F(\frac{e}{D}, \frac{VD\rho}{\mu}). \tag{35}$$

The combination e/D is denoted by ε and is known as the tube's *relative roughness.* The second expression,

$$Re = \frac{\rho D v}{\mu}, \tag{36}$$

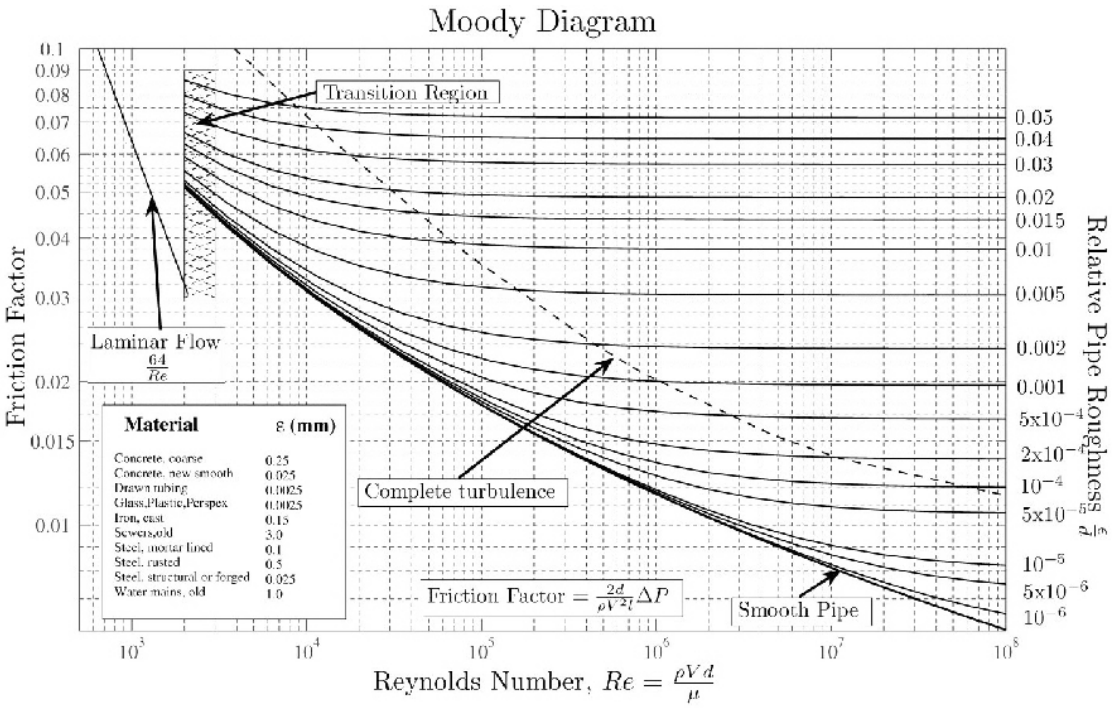

Figure 4: Moody-diagram copied from the *Wikipedia Common* image data base.

is the famous *Reynolds Number*. Just by applying dimensional analysis we have established that

$$P = 2\frac{L\rho V^2}{D} f_F(\varepsilon, Re). \tag{37}$$

(In the literature one will also encounter the friction factor $f_D = 4f_F$ called *Darcy's friction factor*). The friction factor f_F must be determined from more advanced theory and experiments, and in 1944 L.W. Moody presented his the famous diagram which is now called a *Moody diagram*, see Fig. 4. The diagram together with interactive code for calculating f_F are also found in numerous versions on *Internet*.

For those specially interested, one can mention that for very low Reynolds numbers ($Re < 2000$) the flow will be *laminar*. This is the so-called Hagen-Poiseuille flow with a parabolic velocity profile over a smooth tube with circular cross section. For such flow one can show analytically that $P = 32L\mu V/D^2$, *i.e.* $f_F = 16/Re$. For the rest of the chart, there are more or less empirical expressions available. When $Re > 6000$ we have

$$f_F = \max(f_C, f_N), \tag{38}$$

where f_C is the solution of *Colebrook's equation*

$$\frac{1}{f_C^{1/2}} = -1.74 \log\left(\frac{\varepsilon}{3.7} + \frac{1.25}{Re \times f_C^{1/2}}\right), \tag{39}$$

and f_N is given from *Nikuradse's relation* for fully developed turbulent flow,

$$f_N = \frac{4}{(1.14 - 0.87 \log \varepsilon)^2}. \tag{40}$$

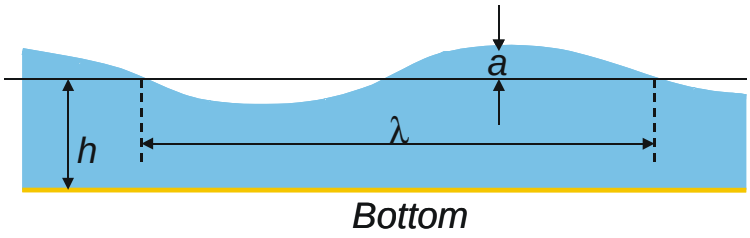

Figure 5: A regular wave on the surface of water.

In the transition between laminar and turbulent flow, the flow is unstable and can switch in an unpredictable way between being laminar or turbulent.

1.3.5 Water Waves

In one space dimension, we can write a regular wave on water surface

$$\eta(x,t) = a\cos(kx - \omega t), \tag{41}$$

where a is the wave amplitude, x the space coordinate, t is time, $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength, $\omega = 2\pi/T$ is the angular frequency and T the wave period, see Fig. 5.

For water waves, k and ω can not be chosen arbitrarily, but must satisfy a *dispersion relation*

$$\omega^2 = f(k,h,a,\cdots). \tag{42}$$

It is reasonable that the angular frequency ω occurs with a second power. Positive and negative frequencies corresponding to waves that move to the right and left, respectively (for positive k). Waves on water may be generated by the wind, boats etc. and are maintained by gravity. For very short waves, $\lambda = O(1\text{cm})$, the surface tension keeps the wave going. The surface tension (or stress), σ is characterized by a surface tension coefficient T (not to be confused with the period) that connects the surface curvature and the tension. In one dimension, the expression is $\sigma = T\partial^2\eta/\partial x^2$. The unit for T is thus

$$[T] = \frac{[\sigma]}{[\partial^2\eta/\partial x^2]} = \left(\frac{\text{kgm}}{\text{s}^2}\frac{1}{\text{m}^2}\right)\frac{\text{m}^2}{\text{m}} = \frac{\text{kg}}{\text{s}^2}. \tag{43}$$

Since gravity is important, the gravitational acceleration g and the water density ρ are also possible parameters in the dispersion relation. We neglect the effect of air motion over the waves. Thus, we end up with the following assumption about the dispersion relation:

$$\omega^2 = f(k,a,h,g,\rho,T). \tag{44}$$

The corresponding dimension matrix is displayed in Table 4. We easily see that the matrix has rank 3, and consequently, there are $7-3=4$ dimensionless combinations. Of several possibilities we choose $\{k,g,\rho\}$ as our core variables. It is not particularly smart to use ω^2, since we do not want ω^2 to enter on the right side of the equation. Furthermore, either ρ or T need to be

	ω^2	k	a	h	g	ρ	T
m	0	-1	1	1	1	-3	0
s	-2	0	0	0	-2	0	-2
kg	0	0	0	0	0	1	1

Table 4: Dimension matrix for the dispersion relation.

involved since these are the only variables that contain kg in their units. It is now easy to find four dimensionless combinations,

$$\begin{aligned} \pi_1 &= \frac{\omega^2}{kg}, \\ \pi_2 &= ak, \\ \pi_3 &= hk, \\ \pi_4 &= \frac{Tk^2}{\rho g}, \end{aligned} \tag{45}$$

and we find

$$\omega^2 = gk\Phi(ak, hk, \frac{Tk^2}{\rho g}). \tag{46}$$

In this formula there are several special cases:

- The wave has very small amplitude compared to the wavelength, $ak \ll 1$
- The water depth is large relative to wavelength, $hk \gg 1$
- The wavelength is much larger than 1cm, $\frac{Tk^2}{\rho g} \ll 1$ (follows from the numerical value of T)

If all three conditions are present, we could write $\omega^2 \approx gk\Phi(0, \infty, 0)$. A more refined analysis (by solving the differential equations for water waves) shows that the relation in this case is $\omega^2 \approx gk$ and that $\Phi(0, \infty, 0) = 1$. If the depth is not so large, we obtain $\omega^2 = gk\Phi(0, hk, 0)$, and a closer analysis here shows that

$$\omega^2 = gk \tanh(hk). \tag{47}$$

If the depth is large, and we have very short waves, only surface tension and not gravity is of importance. A simplified dimensional matrix could then be

	ω^2	k	T
m	0	-1	0
s	-2	0	-2
kg	0	0	1

, (48)

but since only T depends on "kg", it is impossible to combine T with the two others and form a dimensionless combination. Thus, we need another parameter to match "kg" , and the only possibility is ρ. We leave to the reader to show that this gives us

$$\omega^2 = C\frac{Tk^3}{\rho}. \tag{49}$$

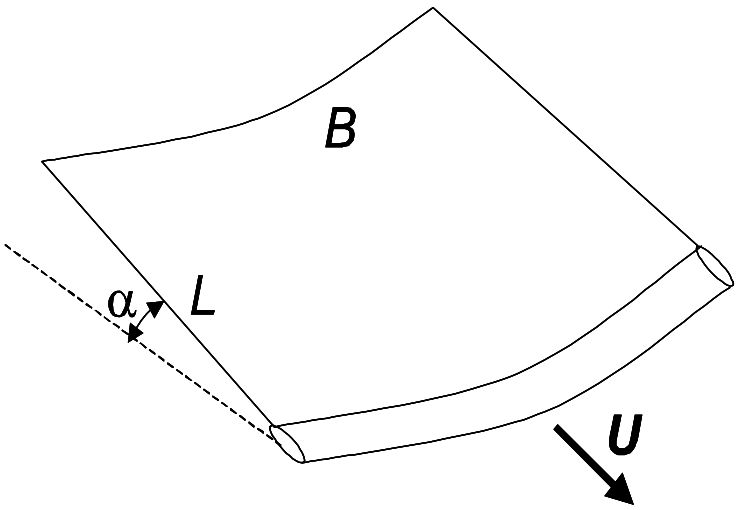

Figure 6: Sketch of the paper airplane in the text.

It turns out that also in this case, $C = 1$. In general, it is possible to show by analytical methods that for $ak \ll 1$, we will have

$$\omega^2 = gk \tanh(kh) \left(1 + \frac{Tk^2}{\rho g}\right). \tag{50}$$

1.3.6 Design of Paper Airplanes

What is the optimal shape of a paper airplane? Even if we restrict ourselves to one kind of models, this is not a simple question and we expect to do a lot of experimentation. Before we start, it may be smart to carry out some dimensional analysis. We shall focus on models that have performed well in *Scientific American*'s paper airplane competitions. The airplanes are made by taking a sheet of paper of length L_0 and width B and fold it with small folds from one side until the center of gravity lies approximately 1/4 from the folded edge, as shown in Fig. 6.

After the folding, the plane has length L. Instead of folding, it is also possible, using a little stiffer paper, to position the center of gravity correctly by using one or more clips in front of the sheet. Ideally, such a wing should slide with constant velocity U in a fixed angle α with the horizontal plane. Assume that one task is to investigate how the speed depends on the length, width, and weight of the airplane. The paper's weight per unit is denoted ρ_p (kg/m^2), the air density ρ_a (kg/m^3), and air viscosity ν (m^2/s).

Let us first consider the friction force F between the airplane and the air. This force must in any case depend on the size and speed of the plane, i.e., L, B and U. Furthermore, we expect that the viscosity of the air is of importance. If we look at L, B, U and ν, we find that none of the units of these variables include kg, and since this occurs in force, we need a few more parameters. It is reasonable to choose ρ_a, while there is no reason why ρ_p should enter the expression for the force. There are other parameters that can be expected to have little impact on the friction force, such as how smooth the paper is, how thick is the fold, etc., but we should have listed the most important ones.

The dimension matrix is constructed similarly to above

	F	B	L	U	ν	ρ_a
kg	1	0	0	0	0	1
m	1	1	1	1	2	0
s	-2	0	0	-1	-1	-3

. (51)

We leave to the reader to show that this gives us three dimensionless variables which can be arranged so that

$$F = L^2U^2\rho_a\Psi\left(\frac{L}{B}, \frac{LU}{\nu}\right). \tag{52}$$

Alternatively, one may write

$$F = LBU^2\rho_a\tilde{\Psi}\left(\frac{L}{B}, \frac{LU}{\nu}\right), \tag{53}$$

and when $B \gg L$, it would be reasonable to replace $\tilde{\Psi}\left(\frac{L}{B}, \frac{LU}{\nu}\right)$ with a function of only one unknown, i.e.

$$\Phi\left(\frac{LU}{\nu}\right) = \tilde{\Psi}\left(0, \frac{LU}{\nu}\right). \tag{54}$$

If the aircraft moves with a constant speed in a fixed angle with the horizontal, the friction force has to balance gravity. Then

$$F = Mg\sin\alpha = \rho_p\left(L_0B\right)\sin\alpha. \tag{55}$$

(In addition, there must be sufficient lift for the airplane to stay in the air). For a wide plane, the speed may then be expressed as

$$U^2 = \frac{\rho_p}{\rho_a}\frac{L_0}{L}g\sin\alpha\Phi\left(\frac{LU}{\nu}\right). \tag{56}$$

The combination LU/ν is again the Reynolds number, and as we see, simple dimensional analysis has given us much insight which we can take with us further in the investigation.

1.4 Summary

In this chapter we have seen, based on two fairly obvious axioms about the nature, that it is possible to derive the quite powerful Buckingham pi-theorem. These axioms are basically laws of nature, of our universe. The theorem is easy to use, but requires that there really is a relationship between the quantities we have listed. In practice, this can be problematic to determine.

If we were asked to find the eigenfrequency ω of a mathematical pendulum, we would assume that this depends on the length of the pendulum's rod (L), the gravitational acceleration (g), the pendulum's position angle from the vertical at the start (α), and the mass of the bob (m). Based on these quantities, there should exist a relationship

$$\Phi(\omega_0, L, g, \alpha, m) = 0. \tag{57}$$

Here we observe, however, that this is impossible, since the mass is the only quantity containing kg in its unit. Either we must remove m, or there must be an additional quantity that we have forgotten. Since is seems impossible to find other reasonable parameters to include, we are forced to remove m. This leads to the following useful observation:

- *Each fundamental unit must occur in at least two of the quantities.*

The standard procedure now gives us

$$\omega_0 = (g/L)^{1/2} f(\alpha). \tag{58}$$

The results of the dimensional analysis are not unambiguous. Instead of writing

$$\Psi(\pi_1, \pi_2, \cdots, \pi_{M-r}) = 0, \tag{59}$$

we could just as well write

$$\begin{aligned} \pi_1 &= f(\pi_2, \cdots, \pi_{M-r}), \\ f(\pi_1, \pi_2) &= g(\pi_3, \cdots, \pi_{M-r}), \\ &\vdots \end{aligned} \tag{60}$$

Here, we use what is appropriate. There is no reason to say that one way of writing the formula is more correct than another. In addition, the dimensionless combinations are not unique. If π_1 is dimensionless, then so is also $\sqrt{\pi_1},\ 1/\pi_1,\ \pi_1^2$. With more experience, one will often recognize common combinations such as Reynolds number, etc.

The core variables was the subset that we used to form the dimensionless combinations. Usually, there are also several possibilities here. If we are interested in finding how a variable (such as R_1) depends on the others, it is reasonable to avoid using R_1 as one of the core variables. In that way, we find a relation of the form $R_1 = \phi(R_2, R_3, \cdots, R_M)$, that is, R_1 does not enter into the arguments in ϕ.

We have treated dimensional analysis as a method to simplify the relationships between physical quantities. Dimensional analysis is used to obtain an overview and can indicate whether we really understand what we are doing.

One of the best properties of dimension analysis is that it gives us a formulation containing the minimum number of free variables. This is in particular valuable for experimental work in the lab or at the computer.

If we decide to find the frequency of a mathematical pendulum by means of experiments only, and assume that $\omega_0 = \Phi(L, g, \alpha, m)$, we may have to determine the function Φ by selecting 10 different values for each variable, that is, perform a total of 10^4 experiments. If we first use dimensional analysis, we realize that it is enough to use only one pendulum, vary the angle α for a reasonable set of values,and then plot α against $\omega_0 (L/g)^{1/2}$ in order to determine the function $f(\alpha)$ in the expression from the dimensional analysis,

$$\omega_0 = \sqrt{\frac{g}{L}} f(\alpha).$$

A similar simplification is also important to do in order to save the number of numerical experiments on a computer, and before setting up experimental plans in statistical experiments.

Dimensional analysis is also crucial when working with *scale models*, that is, doing experiments with models scaled down (or up) in size. Ideally, one would like that the dimensionless combinations are the same for the model as for the original (this is called the *scale laws*).

All comprehensive textbooks on mechanics describes dimensional analysis. For example, both [4] and [3] has nice introductions, while [2] is considered a classic. Moreover, the *Internet* has several million references to *Dimensional Analysis*.

2 SCALING

2.1 Introducing Scaled Variables

After establishing a mathematical model in the form of an equation, it will be necessary to introduce dimensionless variables. Usually it is not difficult to do this, but it can be carried out in several ways, and it is not always easy to see what is most appropriate way. However, there exists an intelligent way of doing this called *scaling the equations*. When the equations are scaled, it is easy to see which parts are important and which are less important. It can be difficult to scale equations, and in any case this will depend on the problem we are considering, even if the equation is basically the same all the time. Somewhat simplified we can say that the scales force us to think about the situation, and in this way we gain insight into what we are doing. The theory in this chapter is mainly taken from [3].

- *To scale a variable u^* means to write the variable as*

$$u^* = Uu, \tag{61}$$

where $[U] = [u^*]$, U is of the same order of magnitude as u^*, and u is of order 1.

Here U is the characteristic size of u^*: If we use U as our unit of measurement, u is neither particularly large nor particularly small. This is a somewhat imprecise definition, but it reflects the fact that scales are not always very well defined.

Until getting used to scaling variables, it is handy to have a notation in order to distinguish between the original variables with units, and the new dimensionless variables. We will do this, as suggested in [3], by attaching * on the original variables, and remove * after the variable has been made dimensionless. After a while, we become tired of writing *, and understand the transition from the context.

Let us consider a variable u^* which is a function of time t^*. It is usually reasonable to use

$$U = \max_{t^*} |u^*(t^*)|$$

as a scale for u^*, even if the minimum of u^* is much smaller. Then, at least $|u| \leq 1$. In practice, this often means to *estimate* the maximum value, since we may not know u^* in detail.

It will also be necessary to find scales for time. Sometimes the maximum value of t^*may be used, but more often the scale is defined as a period over which u^* *varies significantly*. If $u^*(t^*) = \sin at^*$, a reasonable time scale would be $1/a$, since u^* then varies from 0 to 1. As suggested in [3], it is often possible to find a reasonable time scale by looking at (or estimate)

$$\frac{\max |u^*(t^*)|}{\max |du^*/dt^*|}. \tag{62}$$

Such expressions must be used with common sense, and when working with scales, we are not very careful about extra factors such as 2, π, etc. Scaling is not an exact science, – often a rough estimate is all we need.

- *To scale an equation means to introduce dimensionless variables based on the scales of the variables in the equation.*

Depending on the situation we are in, the same equation could be scaled in several ways. After the equation is scaled, it will be clear what are important and less important parts of the equation (if not all are equally important). Often one will be able to get approximate solutions by solving the equation when the less important parts are removed. Knowing the scales of the variables of a mathematical model requires knowledge and physical understanding, and is one of the most important things we do in mathematical modeling. As will be seen below, scaling is not nearly as easy as it sounds. A good example is one of the main modeling examples in [3], where the authors, several years after the book was published, discovered that the time scale they had suggested was not really appropriate (Quite recently, the authors of these notes have suggested a completely different scaling of the same equations).

2.2 Order of Magnitude

We say that the function $f(x)$ is of the *order of magnitude* $g(x)$ when $x \to a$ if there exist two finite numbers $\{m, M\}$, $0 < m < 1 < M$, such that

$$m \leq \frac{f(x)}{g(x)} \leq M \tag{63}$$

for $x \to a$. This is written

$$f(x) = \mathcal{O}(g(x)), \ x \to a, \tag{64}$$

and expressed in words as "$f(x)$ *is of order* $g(x)$ *when* x *is close to* a". Some like to require that $m = \sqrt{1/10}$ and $M = \sqrt{10}$ (what is then $\log_{10} m$ and $\log_{10} M$?), but we prefer a more informal use, *e.g.*

$$\log(1+x) - x = \mathcal{O}\left(x^2\right) \text{ for small } x\text{-s}. \tag{65}$$

For series, the first non-zero term is called the *leading order* term, *i.e.* $4x^3 + 3x^4 + 5x^5 + \cdots$ is *of leading order* x^3 for small x-s.

A slightly different symbol, "$o()$" is more precise: We write

$$f(x) = o(g(x)) \text{ when } x \to a, \tag{66}$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0. \tag{67}$$

Thus,

$$\sin x - x + x^3/6 = \mathcal{O}\left(x^5\right), \tag{68}$$

$$\sin x - x + x^3/6 = o\left(x^4\right), \tag{69}$$

when $x \to 0$.

2.3 A Simple Case Study

In the following artificial and simple example we shall see how scales change depending on the nature of the problem. The example is trivial and easy to solve analytically. The assumption about the friction force is not very realistic.

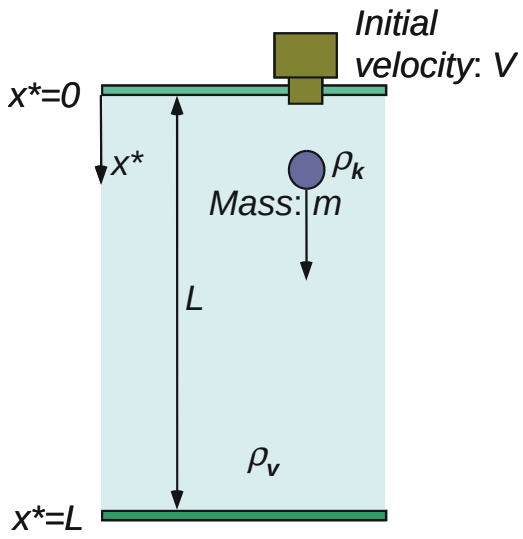

Figure 7: Ball falling or fired into a viscous fluid.

A spherical ball is fired vertically into a viscous fluid as illustrated in Fig. 7. The ball's initial speed is V and the forces acting on the ball is

$$\begin{array}{ll} \text{Gravity:} & gm \\ \text{Friction:} & -k\frac{dx^*}{dt^*} \\ \text{Buoyancy:} & -gm\frac{\rho_v}{\rho_k} \end{array} \tag{70}$$

(Here, ρ_v is the fluid density and ρ_k the density of the ball). The equation of motion follows from Newton's Law and we assume that the ball starts at $x^* = 0$ with velocity V:

$$m\frac{d^2x^*}{dt^{*2}} = gm - k\frac{dx^*}{dt^*} - mg\frac{\rho_v}{\rho_k}, \tag{71}$$

$$x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = V. \tag{72}$$

We shall also assume that $\rho_v < \rho_k$, such that the ball does not eventually float up to the surface, and we replace $g(1 - \frac{\rho_v}{\rho_k})$ with a modified g so that the problem simplifies to

$$\begin{aligned} m\frac{d^2x^*}{dt^{*2}} &= gm - k\frac{dx^*}{dt^*}, \\ x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) &= V. \end{aligned} \tag{73}$$

In this case we can imagine a number of special cases. If the ball had fallen freely with zero initial velocity, it would, at $x = L$, have reached the speed v_{FF} where $v_{FF} = \sqrt{2Lg}$ (Vertical motion under constant acceleration). If, on the other hand, the medium is *very* viscous (think of syrup!), the ball will after a while fall with constant speed v_0 determined by

$$0 = gm - kv_0, \tag{74}$$

i.e. $v_0 = \frac{gm}{k}$.

Below we shall consider three different situations, and it will turn out that the ratio between v_{FF} and v_0 is crucial.

2.3.1 Case A: The friction is large – what happens initially is not very important

This is a situation where either L or the viscosity (here expressed by the constant k) is so great that the ball falls at a constant speed over most of the distance. Assuming that the ball has speed v_0 all the way, we may estimate the time it takes to go from $x^* = 0$ to $x^* = L$ to about

$$T_0 = \frac{L}{v_0} = \frac{Lk}{mg}, \tag{75}$$

and this gives us a reasonable time scale. Depending on the size of V, the actual time the ball uses could be slightly larger or smaller than T_0. There is however an implicit assumption here that V is not very large compared to v_0. The length scale is not a problem, we use L and introduce dimensionless variables x and t as

$$\begin{aligned} x^* &= Lx, \\ t^* &= \frac{Lk}{mg}t. \end{aligned} \tag{76}$$

By bringing this into the equations, we obtain

$$\begin{aligned} m\frac{d^2\left(Lx\right)}{d\left(\frac{Lk}{mg}t\right)^2} + k\frac{d\left(Lx\right)}{d\left(\frac{Lk}{mg}t\right)} &= gm, \\ Lx\left(0\right) = 0, \frac{d\left(Lx\right)}{d\left(\frac{Lk}{mg}t\right)}\left(0\right) &= V, \end{aligned} \tag{77}$$

and after simplification,

$$\begin{aligned} \frac{gm^2}{Lk^2}\frac{d^2x}{dt^2} + \frac{dx}{dt} &= 1, \\ x(0) = 0, \ \frac{dx}{dt}(0) &= \frac{V}{v_0}. \end{aligned} \tag{78}$$

In addition to the variables x and t, the problem contains two dimensionless parameters:

$$\begin{aligned} \varepsilon &= \frac{gm^2}{Lk^2}, \\ \mu &= V/v_0. \end{aligned} \tag{79}$$

We note that

$$\varepsilon = \frac{gm^2}{Lk^2} = 2\frac{1}{2Lg}\left(\frac{gm}{k}\right)^2 = 2\frac{v_0^2}{v_{FF}^2} = 2\left(\frac{v_0}{v_{FF}}\right)^2. \tag{80}$$

Thus, ε is a small parameter (compared to 1) if $v_0 \ll v_{FF}$. It is characteristic for this case that the speed v_0 is much less than the speed the ball would have had at $x^* = L$ if it fell freely. It is typical that when we have scaled the equations, the dimensionless parameters have interesting interpretations that we may apply for *hindsight*.

After the scaling is complete, the equation has the form

$$\begin{aligned} \varepsilon\frac{d^2x}{dt^2} + \frac{dx}{dt} = 1, \ x(0) = 0, \ \frac{dx}{dt}(0) = \mu, \\ \varepsilon = 2\frac{v_0^2}{v_{FF}^2}, \ \mu = \frac{V}{v_0}. \end{aligned}$$

As mentioned above, there is here an assumption that μ is not particularly large. In that case, one might imagine another time scale (see Case C below).

2.3.2 Case B: Small friction. The ball falls approximately freely. V is small compared to v_{FF}.

This problem could have been the same as in Case A, but now with the difference that L is so small that the ball never reaches speeds near v_0. Thus, friction is of little importance.

Again, L is a natural length scale for the x^*. If the ball fell freely and $V = 0$, the ball would fall with nearly constant acceleration, and the time it takes to fall to $x^* = L$ would roughly be $\sqrt{2L/g}$. Since we have already introduced v_{FF}, we apply $T_0 = L/v_{FF}$ as our scale. Certainly, T_0 is only about the half of $\sqrt{2L/g}$, but we do not care about this for a scale estimate. We have already assumed that the speed V is so small that it does not affect the time scale. With these deliberations, we may write

$$\begin{aligned} x^* &= Lx, \\ t^* &= \frac{L}{v_{FF}}t, \end{aligned} \tag{81}$$

and obtain

$$mL\,\frac{2Lg}{L^2}\,\frac{d^2x}{dt^2} \,+\, kL\,\frac{v_{FF}}{L}\,\frac{dx}{dt} \,=\, mg,\; x(0) \,=\, 0,\;\; \frac{dx}{dt}(0) \,=\, \frac{V}{v_{FF}}, \tag{82}$$

and finally

$$2\frac{d^2x}{d\bar{t}^2} \,+\, \varepsilon\frac{dx}{dt}(0) \,=\, 1,\;\; x(0) \,=\, 0,\;\; \frac{dx}{dt}(0) \,=\, \mu, \tag{83}$$

$$\varepsilon \,=\, \frac{v_{FF}}{v_0},\;\; \mu \,=\, \frac{V}{v_{FF}}. \tag{84}$$

Note that the definition of ε has changed compared to Case A, and here, ε is a small parameter if $v_{FF} \ll v_0$. This is a characteristic feature of Case B. The scaling above is only reasonable if V is small compared to v_{FF}. If V is greater than v_{FF}, but still smaller than v_0, the ratio L/V could be a reasonable time scale. We leave to the reader to complete the scaling in this case.

2.3.3 Case C: The ball is released into a highly viscous medium. The initial velocity V is much larger than v_0

In this case, we expect that friction dominates over gravity, and we estimate the length and time scales by looking at the approximate equation

$$m\frac{d^2x^*}{dt^{*2}} = -\,kV,\;\; x^*(0) \,=\, 0,\;\; \frac{dx^*}{dt^*}(0) \,=\, V. \tag{85}$$

If this had been the exact equation, the ball would stop for $t^* = T_0 \,=\, \frac{m}{k}$ (since $\frac{dx^*}{dt^*} \,=\, V - \frac{Vk}{m}t^*$). An associated length scale (where we again disregard a factor of 2) will then be

$$L = VT_0 \,=\, \frac{Vm}{k}. \tag{86}$$

Case	Characteristics	Length scale	Time scale	Equation	Parameters
A	$v_0 \ll v_{FF}$	L	L/v_0	$\varepsilon\ddot{x} + \dot{x} = 1$	$\varepsilon = 2\frac{v_0^2}{v_{FF}}$, $\mu = \frac{V}{v_0}$
B	$v_0 \gg v_{FF}, V < v_{FF}$	L	L/v_{FF}	$2\ddot{x} + \varepsilon\dot{x} = 1$	$\varepsilon = \frac{v_{FF}}{v_0}$, $\mu = v/v_{FF}$
C	$V \gg v_0$	mv/k	m/k	$\ddot{x} + \dot{x} = \varepsilon$	$\varepsilon = \frac{v_0}{v}$, $\mu = 1$

Table 5: A summary of the scaling example

We introduce $x^* = \frac{mV}{k}x$ and $t^* = \frac{m}{k}t$:

$$m\frac{mV}{k}\frac{k^2}{m^2}\frac{d^2x}{dt^2} + k\frac{mV}{k}\frac{k}{m}\frac{dx}{dt} = mg,\ \ x(0) = 0,\ \ \frac{dx}{dt}(0) = 1, \tag{87}$$

which, after some simplification becomes

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = \varepsilon,\ \ x(0) = 0,\ \ \frac{dx}{dt}(0) = 1, \tag{88}$$

$$\varepsilon = \frac{v_0}{V}. \tag{89}$$

In this situation, ε is a small parameter when $V \gg v_0$, and this is the characteristic feature for Case C.

2.3.4 Summary

We have now seen three different situations where weight has been put on various parts of the equation. The problem has, in addition to V, two characteristic speeds, namely $v_{FF} = \sqrt{2gL}$ and $v_0 = \frac{gm}{k}$, and the various situations above are characterized by the mutual size of these speeds. We summarize the results in Table 5.

In all three situations we end up with a parameter ε which is typically small. The related terms in the equation are also small, and by neglecting the terms of order ε, we obtain the simplified equations.

Although it is the rule rather than the exception that we end with terms of different size in a scaled equation, it is also possible that all terms happen to be of the same magnitude.

We leave to the reader to show that the exact solution is

$$x^*(t^*) = \frac{gm}{k}t^* + \left(V - \frac{gm}{k}\right)\frac{m}{k}\left(1 - e^{-t^*/(m/k)}\right), \tag{90}$$

and the graph in Fig. 8 shows how the exact solutions relate to the situations we have seen. Note that the cases we have considered by no means cover the entire chart. By setting $\varepsilon = 0$ for all three situations above, we obtain simplified equations, but Case A is special. With $\varepsilon = 0$, the equation and initial conditions become

$$\dot{x}_0 = 1, \tag{91}$$

$$x_0(0) = 0, \tag{92}$$

$$\dot{x}_0(0) = \mu, \tag{93}$$

and unless μ happens to be 1, it is *impossible* to solve the simplified problem exactly. The general solution to Eq. (91) is

$$x_0(t) = C + t, \tag{94}$$

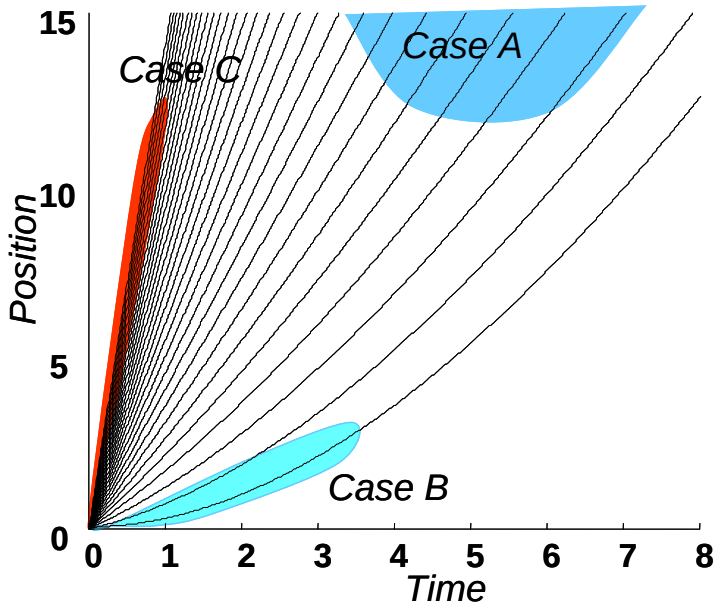

Figure 8: The figure shows the exact paths and an indication of the three situations we have considered.

and since the speed is 1 (v_0 in the original variables), the form is reasonable. The approximate solution is simply not valid near 0, and in order to determine the correct $C = C(\mu)$ a special technique (*singular perturbation*) is required.

In Case B, the approximate equation is

$$\begin{aligned} 2\ddot{x}_0 &= 1, \\ x_0(0) &= 0, \\ \dot{x}_0(0) &= \mu, \end{aligned} \tag{95}$$

which we immediately solve as

$$x_0(t) = t^2/4 + \mu t. \tag{96}$$

We can check the approximate solution by inserting it in the exact equation,

$$2\ddot{x}_0 + \varepsilon\dot{x}_0 - 1 = \varepsilon\left(\frac{t}{2} + \mu\right). \tag{97}$$

The error on the RHS increases with time, and this is reasonable since the approximate solution is not at all limited by friction.

The equation for Case C has approximate solution

$$x_0(t) = 1 - e^{-t}, \tag{98}$$

so here $x_0(t) < 1$ for all t.

Although the solutions in B and C obviously have their weakness, they are great for the situations they are supposed to cover. Convince yourself by drawing the approximate and exact solutions for some choice of ε and μ.

2.4 Scaling Considerations

Arguments based on scale considerations have proven to be quite useful in many contexts, but they require some physical insight and creativity, and are not always so easy to perform.

2.4.1 Turbulence

Fluids are mixed (on the microscopic level) by *molecular diffusion*, and (on the macroscopic level) by *convection*. Molecular diffusion is related to the kinematic viscosity of the fluid (ν, $[\nu] = \mathrm{m}^2/\mathrm{s}$), while convection is the macroscopic motion, typical by visible whirls, observed when we move the spoon around in a cup of tea, or when watching the whirling water in a river, for example.

Suppose we consider a whirl with diameter L. The time scale associated with L and convection with a velocity scale U will be

$$t_K = L/U. \tag{99}$$

The time scale associated with diffusion over a length L may likewise be expressed by the kinematic viscosity, ν and L. The only possibility is

$$t_D = \frac{L^2}{\nu}. \tag{100}$$

We observe that the quotient between these two scales is

$$\frac{t_D}{t_K} = \frac{L^2 U}{\nu L} = \frac{LU}{\nu} = Re, \tag{101}$$

which is a new meaning of the well-known *Reynolds number,* also mentioned above. A Reynolds number $Re \ll 1$ indicates that the mixing is dominated by molecular diffusion, whereas $Re \gg 1$ means that it is dominated by convection.

The value of ν for water is about $10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$. Consider a typical river with width $L = 100\mathrm{m}$, and $U = 1\mathrm{m}$ / s. Then

$$Re \approx \frac{100 \cdot 1}{10^{-6}} = 10^8, \tag{102}$$

and the mixing of the water in the river is entirely dominated by convection.

In turbulent flow large vortices initiate motion of small vortices which, in turn, set into motion (and keep alive) even smaller vortices, and so on. In the very small vortices viscosity will reduce the motion and the kinetic energy is eventually transferred into heat. The kinetic energy dissipation (loss of energy) is mainly from these small vortices with a length scale l' and velocity scales u'. We can estimate the energy loss (E) per. time and unit mass by assuming that $E = E(l', u', \nu)$, and that $E \propto u'^2$ (in other words, proportional to the kinetic energy that is present in the smallest vortices). Simple dimensional analysis then gives

$$E \propto \nu \left(\frac{u'}{l'}\right)^2, \tag{103}$$

and an estimate for E will be $E = \nu \left(\frac{u'}{l'}\right)^2$. From the above we can further assume that the smallest vortices have $Re \approx 1$, or $t_K \approx t_D$, that is

$$\frac{l'u'}{\nu} = 1. \tag{104}$$

Thus,

$$l' = \left(\frac{\nu^3}{E}\right)^{1/4}, \qquad (105)$$
$$u' = (\nu E)^{1/4}.$$

These scales are called the *Kolmogorov's micro scales* in turbulence theory. These are the smallest scales that occur before the diffusion takes over and turns the kinetic energy over to heat by internal friction.

If we mix 1kg of water with a mixer with an output of 100W, this power would disappear in the smallest vortices, and consequently the diameter of these vortices is of the order

$$l' = \left(\left(\frac{10^{-6}\mathrm{m}^2}{\mathrm{s}}\right)^3 / \left(100\frac{\mathrm{kgm}^2}{\mathrm{s}^2\cdot\mathrm{s}\cdot 1\mathrm{kg}}\right)\right)^{1/4} = 10^{-5}\mathrm{m} = 0.01\mathrm{mm}. \qquad (106)$$

2.4.2 Geometric Similarity of Animals

Why do we look like we do? It has long been known that animal forms are not just random, but a result of the strength of muscles and bones in relation to the strength of gravity here on Earth. If we could reduce a human to Thumbelina-size, it turns out that the body would immediately be worn to pieces by the muscles. Therefore, insects usually have very small muscles (thin legs!) in relation to there size.

The discussion below is taken from the note *Dimensional Analysis* of Professor Kristian B. Dysthe, University of Tromsø, 1992. One of his references is the world famous book *On Growth and Form* by D'Arcy W. Thompson, first published in 1917.

We shall first look at animals approximately geometrically similar, and having a typical length scale L. We may then argue that their

1. weight is proportional to their volume, that is, $\propto L^3$
2. muscle power is proportional to the amount of muscle fibers, which in turn is proportional to the muscle cross-sectional area, $\propto L^2$
3. ability to do work (and produce heat), power $\propto$ lung capacity $\propto$ oxygen uptake $\propto$ surface of the lungs $\propto L^2$ (may be somewhat questionable because of the fractal structure of the lung)

Jumping. When an animal wants to jump into the air, it must produce a certain amount of energy which becomes its kinetic energy the moment it leaves the ground. The energy is produced by accelerating the body over a distance, $\mathcal{O}(L)$, multiplied by the power it generates, $\mathcal{O}(L^2)$. In other words, the supply of kinetic energy = force $\times$ distance $\propto \mathcal{O}(L^2) \times \mathcal{O}(L) = \mathcal{O}(L^3)$. The necessary potential energy for a jump of height H will likewise be $H = Hmg \propto HL^3$, where m is the mass of the animal. This therefore gives

$$HL^3 = \text{const.} \cdot L^3, \qquad (107)$$

or that H is constant. Thus, we get the somewhat surprising result that all animals of the same shape jump equally high!

It should also be similar to jump down a certain height H and land in a controlled manner. Cats seem to have this property, and otherwise it is alleged that the kangaroo and a jumping mouse (which by their names would have slightly different size!) can jump equally high.

Running uphill. From the observation above, the power an animal manages to maintain will be proportional to L^2, and since the required power to keep a speed v up a hill with a slope angle α is

$$(mg \sin \alpha) \cdot v \propto L^3 v, \tag{108}$$

we obtain

$$v L^3 = \text{const.} \times L^2, \tag{109}$$

or

$$v \propto 1/L. \tag{110}$$

Small animals can therefore keep a higher speed than large animals when running uphill.

Diving Animals. Assume that all animals during a dive are moving at speed v. The friction force that must be overcome will typically be proportional to the square of the velocity and cross sectional area of the animal, i.e. $F \propto v^2 L^2$ (This may be concluded from formulas derived from dimensional analysis). The total energy consumed within the water is $F \cdot (v \cdot t_{\max})$, where $t_{\max}$ denotes the maximum time it can stay under water. Since the energy stored in the animal will be proportional to L^3, then

$$L^2 \cdot t_{\max} \propto L^3, \tag{111}$$

or

$$t_{\max} \propto L. \tag{112}$$

This means, in other words, that large animals can stay longer under water than small animals, and this we know from the marine mammals.

We leave to the reader to speculate about any other scaling arguments, for example, what the Mars and Jupiter residents look like.

Finally, we shall consider two examples from sport.

Weightlifting. For equally shaped weight lifters, the muscle strength is proportional to L^2 and the weight is proportional to L^3. In other words, the force should be proportional to the weight of power 2/3. Figure 9 shows that this holds astonishingly well.

Rowing. We consider boats with the same shape and the typical length L. The necessary input power (due to friction and wave resistance, called *drag*) to maintain a speed $v_{\max}$ is the same here as for diving animals,

$$v_{\max} \cdot F \propto v_{\max}^3 L^2. \tag{113}$$

The available power is proportional to the number of rowers, N and the displaced fluid volume of the boat is also proportional to N or L^3. Thus

$$L \propto N^{1/3}, \tag{114}$$

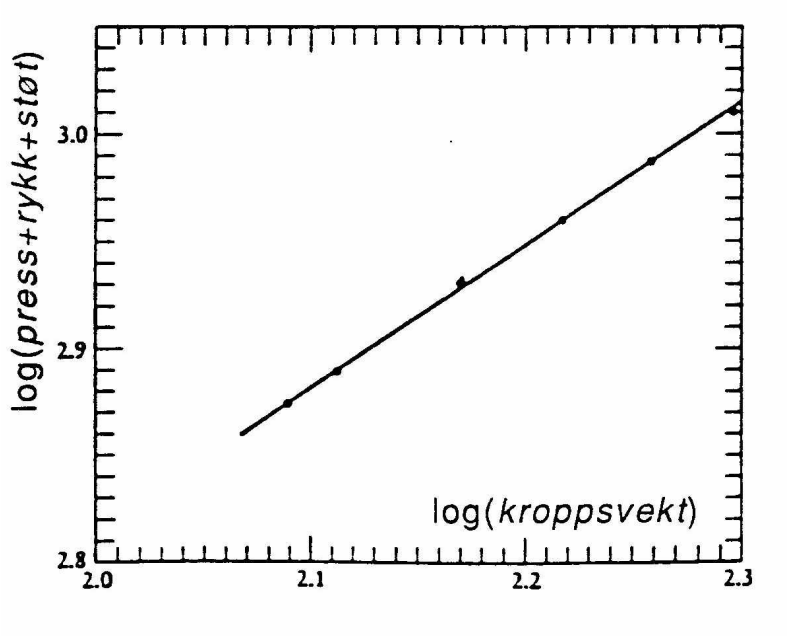

Figure 9: The world records in Snatch + Jerk + Press as a function of the lifter's body weight (kroppsvekt). Note that the line has slope 2/3 (The origin of the figure is unknown).

and we obtain

$$v_{\max}^3 L^2 \propto N, \tag{115}$$

or

$$v_{\max} \propto N^{1/9}. \tag{116}$$

It is surprising that the rate increases so slowly with the number of rowers, but Fig. 10 shows that this indeed seems to be the case.

3 REGULAR PERTURBATION

In this section we shall consider a way to handle equations containing small parameters, and the scaled equations from Case B and Case C in the case study in Sec. 2.3 are of this form. The basic idea is to write the solution as a power series in the small parameter and determine the terms in the series recursively. We shall take a closer look at this methodology and show how it works on some simple examples. Regular perturbation is one of the most common techniques in traditional applied mathematics, and is well treated in several textbooks. The presentation below is incomplete, but adapted to what we are going to need.

We have a *regular perturbation problem* if we have an equation

$$D(x, \cdots, \varepsilon) = 0, \tag{117}$$

containing a small parameter ε so that the full solution, x_{sol}, approaches solution x_0 of the reduced equation

$$D(x, \cdots, 0) = 0, \tag{118}$$

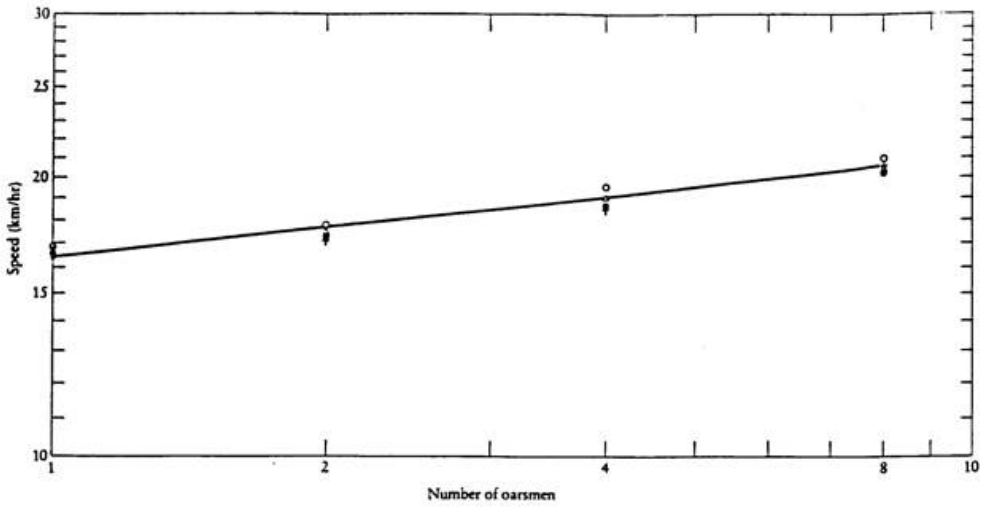

Figure 10: Speed as a function of the number of rowers. The line has slope equal to 1/9. The origin of the graph is unknown.

when ε tends to 0. The statement is somewhat imprecise, as we say nothing about how x_{sol} approaches x_0. If we then know that ε is small (after the equation is scaled), we may approximate the complete solution x_{sol} with x_0. This is pursued further by writing the solution in the form of a power series in ε,

$$x_{sol} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots . \tag{119}$$

for then to come up with a sequence of simpler equations for x_0, x_1, $\cdots$. Since ε is small, we expect that the terms in the series become smaller and smaller, and that the approximation gets better the more terms we include. In practice, it is not that easy. The solution of the equations for x_i often gets more complicated as i increases, and power series do not tend to have very impressive convergence properties.

If we forget about these objections, the method of regular perturbation is easy to state:

1. Write the solution as a power series in ε,

$$x_{sol} = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^2 x_3 + \cdots \tag{120}$$

2. Put the series into the equation and clean up the expression so that we obtain a new power series in ε,

$$\begin{aligned} D(x_{sol}, \varepsilon) &= D(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^2 x_3 + \cdots, \varepsilon), \\ &= P(x_0, 0) + P_1(x_0, x_1)\varepsilon + P_2(x_0, x_1, x_2)\varepsilon^2 + \cdots \end{aligned} \tag{121}$$

3. Set each coefficient in the series equal to 0 and solve the equations you then get *recursively*:

$$\begin{aligned} P_0\,(x_0, 0) &= 0, \\ P_1(x_0, x_1) &= 0, \\ P_2(x_0, x_1, x_2) &= 0, \\ &\cdots \end{aligned} \tag{122}$$

This method gives us $x_0,\ x_1,\ x_2, \cdots$, and the idea may be used in many connections:

- For approximate solutions to algebraic and transcendental equations
- For approximate expressions to integrals
- For ordinary and partial differential equations

Perturbation analysis is often complementary to numerical techniques. In many situations, numerical methods have problems when ε is small (this is especially the case for *singular perturbation* discussed later). The perturbation analysis gives us the asymptotic relations which are useful when ε goes to 0, in contrast to a small number of numerical calculations where we need to keep ε fixed for each calculation. In other contexts there is no really small (or large) parameter to use, and there is no way around numerical calculations.

Perturbation analysis had its best days before we had computers with opportunities for large scale numerical calculations. In particular in the field of aerodynamics and other fluid mechanics. perturbation analysis has been widely recognized. Today, there are computer programs for symbolic manipulation that enables us to find perturbation solutions of orders we could only dream about. However, sometimes profit is marginal, – if one does not achieve reasonable approximations with one or two terms, there is often little to gain by calculating more terms.

3.1 The Projectile Problem

The Projectile Problem, discussed in the book of Lin & Segel [3], pp. 233, is a simple and instructive example of how regular perturbation works. The problem leads to a non-linear differential equation where it is not possible to write the solution in explicit form using elementary functions.

3.1.1 The Model

A projectile is sent vertically up from a planet without atmosphere. The motion is described by the position $x^*\,(t^*)$ above the planet's surface, where t^* is the time and

$$x^*\,(0) = 0,\ \frac{dx^*}{dt^*}\,(0) = V. \tag{123}$$

The projectile will be affected by a force given by Newton's law of gravitation,

$$F\,(x^*) = -G\frac{Mm}{(R+x^*)^2}, \tag{124}$$

where G is the gravitational constant, M is the planet's mass, R the planet's radius, and m the projectile's mass. Similar to Earth, the gravity force on the planet's surface may be written as $F\,(0) = -mg$, so that $g = GM/R^2$. Thus, it follows that

$$m\frac{d^2x^*}{dt^{*2}} = -\frac{R^2gm}{(R+x^*)^2}. \tag{125}$$

The mathematical model thus consists of the non-linear differential equation

$$\frac{d^2x^*}{dt^{*2}} = -\frac{R^2g}{(R+x^*)^2}, \tag{126}$$

with the initial conditions stated in Eq. 123.

3.1.2 Scaling

We are going to study a situation where V is much smaller than the planet's *escape velocity*. If V is larger than the escape velocity, the projective will leave the planet permanently. For Earth the escape velocity is about 11.2km/s. However, here the assumption implies that $x^*(t^*) \ll R$ for the whole trip of the projectile. Under this assumption, the equation simplifies to

$$\frac{d^2x^*}{dt^{*2}} = -\frac{R^2 g}{(R+x^*)^2} = -\frac{g}{(1+x^*/R)^2} \approx -g. \tag{127}$$

This equation may easily be solved with the given initial conditions:

$$x^*(t^*) \approx -\frac{1}{2}gt^{*2} + Vt^*. \tag{128}$$

The approximate maximum height follows from Eq. 128 by observing that the time to maximum height is approximately given by

$$\frac{dx^*}{dt^*} \approx -gt^* + V = 0, \tag{129}$$

or

$$t_{\max} \approx \frac{V}{g}. \tag{130}$$

Thus,

$$x_{\max} \approx -\frac{1}{2}g\left(\frac{V}{g}\right)^2 + V\left(\frac{V}{g}\right) = \frac{1}{2}\frac{V^2}{g}. \tag{131}$$

Reasonable scales for Eq. 125, where we do not care about factors of 2, will now be

$$X = \frac{V^2}{g},\ T = \frac{V}{g}. \tag{132}$$

Inserted into the equation, this leads to

$$\frac{d^2\left(\frac{V^2}{g}x\right)}{d\left(\frac{V}{g}t\right)^2} = -\frac{R^2 g}{\left(R+\frac{V^2}{g}x\right)^2}, \tag{133}$$

$$\frac{V^2}{g}x(0) = 0,\ \frac{d\left(\frac{V^2}{g}x\right)}{d\left(\frac{V}{g}t\right)} = V, \tag{134}$$

and after cleaning up, we have the scaled equation:

$$\ddot{x} = -\frac{1}{(1+\varepsilon x)^2}, \tag{135}$$

$$x(0) = 0,\ \dot{x}(0) = 1,\ \varepsilon = \frac{V^2}{Rg}. \tag{136}$$

It turns out that it is not possible to express the solution of this equation, $x = x(t, \varepsilon)$, by means of elementary functions (this is not quite obvious!). Note that since the parameter ε is approximately equal to $2x_{\max}/R$, it is indeed small under the assumption we made above.

3.1.3 Solution by Means of Regular Perturbation

We shall solve the equation 135 using regular perturbation according to the recipe above, and we start by putting

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{137}$$

into the equation:

$$\begin{aligned}
\ddot{x} &= \ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \cdots = -(1+\varepsilon x)^{-2} \\
&= -\left[1 + (-2)\varepsilon x + \frac{(-2)(-3)}{2}(\varepsilon x)^2 + \cdots\right] \\
&= -1 + 2\varepsilon(x_0 + \varepsilon x_1 + ...) - 3\varepsilon^2 x_0^2 + \cdots \\
&= -1 + \varepsilon 2x_0 + \varepsilon^2(2x_1 - 3x_0^2) + \cdots
\end{aligned} \tag{138}$$

(Note the use of Newton's binomial theorem). By collecting the coefficients in front of each power of ε, we find the system

$$\begin{aligned}
&\ddot{x}_0 = -1, \\
&\ddot{x}_1 = 2x_0, \\
&\ddot{x}_2 = 2x_1 - 3x_0^2, \\
&\ddot{x}_3 = 2x_2 + 2x_0x_1 - 2x_0\left(2x_1 + x_0^2\right) - 2\left(2x_1 - 3x_0^2\right)x_0, \\
&\quad \cdots
\end{aligned} \tag{139}$$

To find the last equation, we had to expand the series in Eq. 137 to order ε^3. We must also decide what to do with the initial conditions, but here it is reasonable to use

$$\begin{aligned}
&x_0(0) = 0,\ \dot{x}_0(0) = 1, \\
&x_1(0) = 0,\ \dot{x}_1(0) = 0, \\
&x_2(0) = 0,\ \dot{x}_2(0) = 0, \\
&\qquad \cdots
\end{aligned} \tag{140}$$

Thus, x_0 takes care of the initial conditions, which are consequently satisfied no matter where we stop the series expansion. The solution for x_0 follows immediately from equation 139:

$$x_0(t) = t - \frac{1}{2}t^2, \tag{141}$$

and by introducing this into the next equation, in 139, we find

$$\ddot{x}_1 = 2x_0 = 2\left(t - \frac{1}{2}t^2\right), \tag{142}$$

or

$$x_1(t) = \frac{1}{3}t^3 - \frac{1}{12}t^4. \tag{143}$$

Note that only the particular solutions change with every step, and that the contribution from the homogeneous solutions disappear for $x_i(t)$ when $i \geq 1$.

Commonly, the algebra quickly becomes quite complicated, but today we can make good use of software for symbolic manipulation, such as *Maple*, *MuPad*, *Mathematica,* or the free *wxMaxima* (*Maple* was used here):

$$x(t) = t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right) + \mathcal{O}\left(\varepsilon^3\right). \tag{144}$$

From this solution, we also find a more accurate equation for the time to the maximum,

$$\begin{aligned}&\frac{d}{dt}\left(t - \frac{1}{2}t^2 + \varepsilon\left(\frac{1}{3}t^3 - \frac{1}{12}t^4\right) + \varepsilon^2\left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right)\right)\\ &= 1 - t + \varepsilon t^2 - \frac{1}{3}\varepsilon t^3 - \varepsilon^2 t^3 + \frac{11}{12}\varepsilon^2 t^4 - \frac{11}{60}\varepsilon^2 t^5 = 0.\end{aligned} \tag{145}$$

This is a fifth-degree equation, but since we expect the solution t_m to be close to 1, we can, in accordance with the foregoing, try a perturbation expansion:

$$t_m = 1 + a\varepsilon + b\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{146}$$

By introducing this into Eq. 145, we find

$$\begin{aligned}0 &= 1 - \left(1 + a\varepsilon + b\varepsilon^2\right) + \varepsilon\left(1 + a\varepsilon + b\varepsilon^2\right)^2 - \frac{1}{3}\varepsilon\left(1 + a\varepsilon + b\varepsilon^2\right)^3 - \varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^3 +\\ &+ \frac{11}{12}\varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^4 - \frac{11}{60}\varepsilon^2\left(1 + a\varepsilon + b\varepsilon^2\right)^5 + \cdots\\ &= \left(-a + \frac{2}{3}\right)\varepsilon + \left(a - \frac{4}{15} - b\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right).\end{aligned} \tag{147}$$

This gives to $\mathcal{O}\left(\varepsilon^2\right)$

$$a = \frac{2}{3},\ b = a - \frac{4}{15} = \frac{2}{5}, \tag{148}$$

and

$$t_m = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{149}$$

It is reasonable that the time it takes up to a maximum increases a little from 1, since gravity acting on the projectile becomes weaker as it rises. The figure 11 shows some numerical solutions created using $Matlab^{TM}$.

The perturbation expansion to zeroth, first and second order are compared to the numerical solution on the figures 12 and 13.

3.1.4 Analytical Solution

As remarked above, is not possible to express the full solution of

$$\ddot{x} = -\frac{1}{(1+\varepsilon x)^2},\ \ x(0) = 0,\ \ \dot{x}(0) = 1, \tag{150}$$

in closed form by means of elementary functions. However, it is possible to do something. After multiplying the equation with $\dot{x}$,we find

$$\frac{d}{dt}\left(\dot{x}^2/2\right) = \frac{d}{dt}\left(\frac{1}{\varepsilon}\frac{1}{1+\varepsilon x}\right), \tag{151}$$

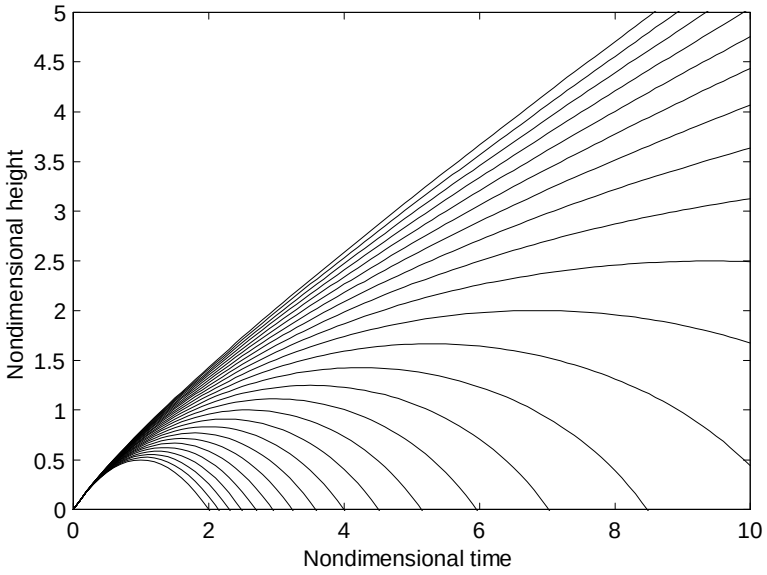

Figure 11: Numerical solutions shown for $\varepsilon = 0\,(0.1)\,3$. When $\varepsilon \geq 2$, the initial speed is above the escape speed, and the projectile never returns to $x = 0$.

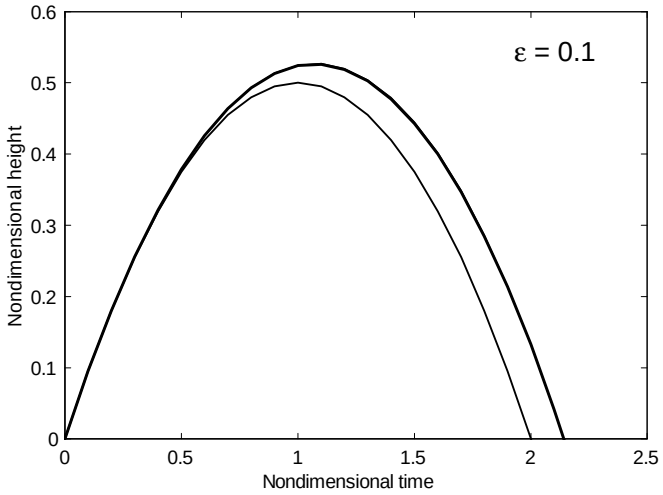

Figure 12: Numerical solution and perturbation solutions for $\varepsilon = 0.1$. The numerical solution (thick line) and the perturbation solutions to first and second order collapse on the graph. The solution for x_1 is different, however.

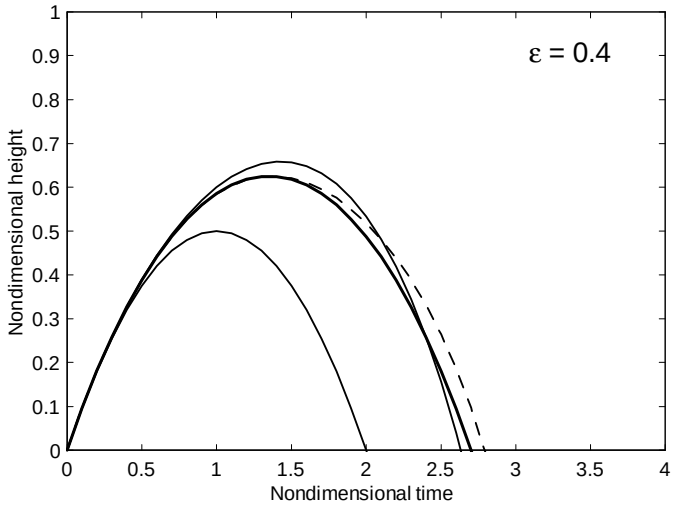

Figure 13: Similar to previous figure for $\varepsilon = 0.4$. Numerical solution: thick curve; x_0 and $x_0 + \varepsilon x_1$: thin curve; $x_0 + \varepsilon x_1 + \varepsilon^2 x_2$: dashed curve.

or that

$$\frac{\dot{x}^2}{2} - \frac{1}{\varepsilon}\frac{1}{1+\varepsilon x} = \text{ constant.} \tag{152}$$

This tells us that the motion is conservative, such that the sum of the potential and kinetic energies is constant. By introducing the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$, we find that the constant is equal to $1/2 - 1/\varepsilon$. This leads to a first order non-linear equation:

$$\dot{x}^2 = \frac{1 + (\varepsilon - 2)x}{1 + \varepsilon x}, \quad x(0) = 0. \tag{153}$$

If $\varepsilon < 2$, $\dot{x}$ will be 0 for

$$x_{\max} = \frac{1}{2 - \varepsilon}. \tag{154}$$

This is therefore the exact expression for the maximum height of the projectile when $\varepsilon < 2$. If $\varepsilon > 2$, the speed will always be greater than 0 and the projectile continues to the boundaries of the universe. Note that for Earth, $\varepsilon = 2$ means that

$$V = \sqrt{2Rg} = \sqrt{2 \times \frac{40000000}{2\pi} \times 9.81 \frac{\text{m}}{\text{s}}} \approx 11.17\text{km/s}, \tag{155}$$

which is therefore the escape velocity referred to above.

By taking the square root of Eq. 153 and separating the variables, and as long as $\dot{x} \geq 0$, we may write the solution of Eq. 153, implicitly as

$$t = \int_0^x \sqrt{\frac{1 + \varepsilon s}{1 + (\varepsilon - 2)s}} ds. \tag{156}$$

This integral turns out to be solvable,

$$\int \sqrt{\frac{a+s}{b-s}} ds = \frac{a+b}{2} \arcsin\left(\frac{2s + a - b}{a+b}\right) - \sqrt{(a+s)(b-s)} + C = F(s,a,b) + C. \tag{157}$$

Thus,

$$t = \int_0^x \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}}ds = \sqrt{\frac{\varepsilon}{2-\varepsilon}}\int_0^x \sqrt{\frac{1/\varepsilon+\varepsilon s}{1/(2-\varepsilon)-s}}ds$$
$$= \sqrt{\frac{\varepsilon}{2-\varepsilon}}\left[F(x,\frac{1}{\varepsilon},\frac{1}{2-\varepsilon})-F(0,\frac{1}{\varepsilon},\frac{1}{2-\varepsilon})\right]. \tag{158}$$

Since we already know that

$$x_{\max} = \frac{1}{2-\varepsilon}, \tag{159}$$

when $\varepsilon < 2$, we also find an exact expression for t_m:

$$t_m = \int_{s=0}^{1/(2-\varepsilon)} \sqrt{\frac{1+\varepsilon s}{1-(2-\varepsilon)s}}ds = \frac{\frac{\pi}{2}-\arcsin(1-\varepsilon)+\sqrt{(2-\varepsilon)\varepsilon}}{(2-\varepsilon)^{3/2}\varepsilon^{1/2}}$$
$$= 1+\frac{2}{3}\varepsilon+\frac{2}{5}\varepsilon^2+\frac{8}{35}\varepsilon^3+\mathcal{O}\left(\varepsilon^4\right). \tag{160}$$

The start of the power series is similar to what we found in Eq. 149.

3.2 Florence Griffith Joyner and the World Record in 100 meters

Florence Griffith Joyner, "Flo-Jo" (1959–98) was an American track runner who is still (as of this writing) the holder of the official world record in 100 meters, 10.49s. The record was set during a quarter-final of the US qualifying heats for the Seoul Olympics in 1988. The wind gauge registered 0 m/s, while many argued that there was considerable tailwind, estimated to about 4 m/s, and that the meter did not work. In the rest of the qualifying races she ran on times around 10.7s.

A sprinter is dependent on the pushing force she/he is able to produce. This force may be written Mp^*, where M is the runner's mass and p^* a parameter with the unit of acceleration. The maximum pushing force is thus MP, where P is the maximum p^* the runner is able to produce.

In addition, there are two forces slowing the sprinter: *air resistance* and *internal friction*. The internal friction, which represents the resistance of muscles and joints, is believed to be written in the form Mu^*/τ, where u^* is the runner's speed and τ is a characteristic time constant. Measurements of different runners, including Ben Johnson and Carl Lewis, have given $P \approx 10\text{m/s}^2$ and $\tau \approx 1\text{s}$, and we shall use these values below.

Based on Newton's second law, we can now write the equation of motion for the sprinter

$$M\frac{du^*}{dt^*} = Mp^*(t^*) - M\frac{u^*}{\tau} - F_l, \tag{161}$$

where F_1 represents the air resistance. The expression for the air resistance may be found by dimensional analysis. It is reasonable to assume that F_l depends on the air density, ρ_{air}, airs kinematic viscosity ν, the runner's velocity u^*, and the runner's cross-sectional area A in the direction of motion. In addition, we need a length scale L, for which we may use $\sqrt{A}$. It is left to the reader to show that from dimensional analysis we may now write

$$F_l = \frac{1}{2}\rho_{luft}C_D\left(Re\right)Au^{*2}, \tag{162}$$

where C_D is an unknown function, the so-called *drag coefficient*, which depends on *Reynolds number*, $Re = \frac{\sqrt{A}u^*}{\nu}$.

Let us now scale Eq. 161. Since we already know that τ is a typical time constant, we decide to use this as our time scale. We know the maximum P, and because of its unit, this is a natural scale for the acceleration. Thus, we scale the velocity by using $P\tau$ and obtain

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t), \tag{163}$$

where

$$\varepsilon = \tfrac{1}{2}\rho_l C_D \left(\frac{\sqrt{A}u^*}{\nu}\right) \tau^2 P \tfrac{A}{M}. \tag{164}$$

Now C_D itself is depending on the speed, but measurements of air resistance for irregular bodies have shown that C_D is almost constant for $2 \times 10^4 < Re < 10^6$, which covers mainly what we are facing here. This value of C_D is close to 1, which we for simplicity shall use below. With P and τ given as above, $A \approx 0.45\text{m}^2$ and $\rho_{air} \approx 1.2\text{kg/m}$, ε is thus about 0.035 for an athlete weighing 70 to 80 kg. Since $|u| \leq 1$, we conclude that the air resistance is a relatively small term in the equation.

In order to solve 163 by regular perturbation, we write

$$u(t) = u_0(t) + \varepsilon u_1(t) + \mathcal{O}(\varepsilon^2) \tag{165}$$

and put this in 163:

$$\dot{u}_0 + \varepsilon\dot{u}_1 + \mathcal{O}(\varepsilon^2) + u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2) + \varepsilon(u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2))^2 = p. \tag{166}$$

We collect all parts of the same order in ε and use $u(0) = 0$ as the initial condition. This gives us a sequence of first order equations:

$$\begin{aligned} &\dot{u}_0 + u_0 = p,\ u_0(0) = 0, \\ &\dot{u}_1 + u_1 = -u_0^2,\ u_1(0) = 0, \\ &\dot{u}_2 + u_2 = -2u_0u_1,\ u_2(0) = 0, \\ &\qquad\vdots \end{aligned} \tag{167}$$

In order to solve the equations, we must also decide what to use for the acceleration $p(t)$. Let us for simplicity assume that $p^*(t) = P$, that is, $p(t) \equiv 1$. The first two terms of perturbation expansion are thus determined by

$$\begin{aligned} &\dot{u}_0 + u_0 = 1,\ u_0(0) = 0, \\ &\dot{u}_1 + u_1 = -u_0^2,\ u_1(0) = 0, \end{aligned} \tag{168}$$

and we easily find that the solution to order ε is

$$u(t) = 1 - e^{-t} + \varepsilon[-1 + 2te^{-t} + e^{-2t}] + \mathcal{O}\left(\varepsilon^2\right). \tag{169}$$

By plotting the graph we see that the sprinter reaches maximum velocity ($u_0(t) \approx 1$) approximately at $t = 3$, that is, after $3\tau = 3$ seconds.

Let us now assume that we have a winds blowing parallel to the running direction. This leads to a modified drag

$$F_l = \tfrac{1}{2}\rho_l C_D A(u^* - W)^2. \tag{170}$$

Wind speed is scaled similar to the runner's speed, so that in dimensionless variables the dimensionless wind is given by $\delta = W/(\tau P)$.

Let us determine the maximum velocity U a sprinter can hold as a function of δ to the first order in ε. The maximum speed is achieved when the acceleration is zero, i.e., given by the equation

$$U + \varepsilon(U - \delta)^2 = 1. \tag{171}$$

Check that the solution to first order in ε is

$$U = 1 - \varepsilon(1 - \delta)^2 + \mathcal{O}(\varepsilon^2). \tag{172}$$

We assume that Florence performed her maximum in all races and (somewhat unrealistically) that she had the maximum speed throughout the race. The maximum speed without wind is $U_0 = 1 - \varepsilon$, while $U_4 = 1 - 0.36\varepsilon$ with a tailwind equal to 4m/s, that is, $\delta = 0.4$. The total time used for 100m will be $T = (100\text{m})/U$, and thus

$$T_0/T_4 = (1 - 0.36\varepsilon)/(1 - \varepsilon). \tag{173}$$

To get an idea of what this means in time, we must find "her" ε. We assume that she has the same acceleration, cross-sectional area and drag coefficient as the one above. Her weight, however, should be somewhat less, let's say 60 kg. That gives $\varepsilon \approx 0.04$. Expressed in time 4 m/s tailwind gives about

$$10.7\text{s} \cdot \frac{1 - 0.04}{1 - 0.04 \times 0.36} = 10.42\text{s}. \tag{174}$$

As she accelerated the first three seconds, this fits very well with the time she actually used. It should also be noted that she ran on 10.54s in the finals, but then the tailwind was about 2m/s (Was that reasonable?).

3.3 Modeling the Kidney Function

This modeling example is also thoroughly covered in [3], Chapter 8, The example deals with the whole process of physical understanding, formulation of mathematical model, scaling, and finally regular perturbation. In the modeling we meet conservation principles that will also be important later in the course.

After observations of kidney tissue, J. Diamond suggested in 1967 that salt and water are expelled from the kidneys in an indirect way by means of *secondary channels*. Such channels have been found in the walls of the main channels, and it is speculated that this might explain how kidneys work. The proposed mechanism is outlined in Figure 14. At the inner end of the secondary channels there are chemical pumps that send salt into the channel with the consumption of chemical energy. Consequently, the solution in the channel has higher salt concentration than the surrounding tissue and the main channel. This means that there is a gradient in the salt concentration towards the opening. Concentration difference between the channel and the

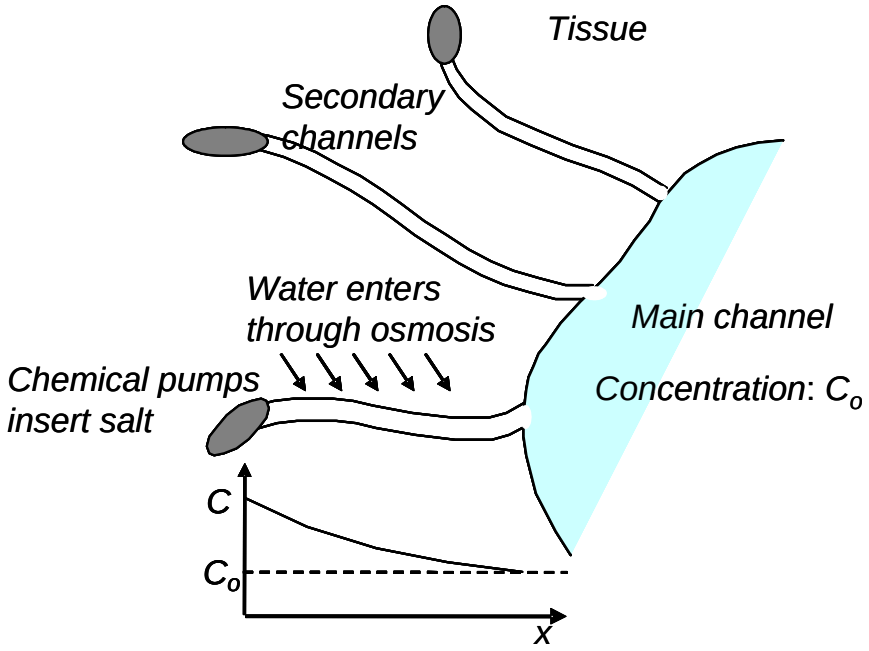

Figure 14: Sketch showing how one thinks that the kidneys are functioning.

surrounding tissue causes the water to enter the channel through the walls because of *osmosis*. The water flowing into the main channel transports at the same time the salt out of the channel. Under stationary conditions we get a so-called *standing gradient* in the salt solution in the channel. Actually, salt is transported in the channel both by *diffusion* and by *convective* transport by the water flow. The problem we are facing, is to set up a mathematical model for a secondary channel and determine how the salt concentration and water flow vary in the channel, and, in particular, how effective the proposed mechanism might be.

The description above includes a number of concepts that we first need to consider. Molecular *osmosis* is an important mechanism in biological systems. Osmosis involves transport through membranes that are so finely meshed that they do not allow large molecules to pass through the channel. In our case the channel wall is modeled as an osmotic membrane, sketched in Fig. 15.

If the ion concentrations on the sides of the membrane are C_1 and C_2 respectively, the net amount of water passing through the membrane per area and unit time, J, may be written as

$$J = P(C_2 - C_1). \tag{175}$$

The constant P is called *permeability*. The units of the variables in this equation are as follows:

$$\begin{aligned} [J] &= \frac{\text{Volume}}{\text{Area} \times \text{Time}} = \frac{\text{m}^3}{\text{m}^2\text{s}} = \frac{\text{m}}{\text{s}}, \\ [C] &= \frac{\#\text{ ions}}{\text{Volume}} = \frac{\text{osmol}}{\text{m}^3}, \\ [P] &= \frac{\text{m/s}}{\text{osmol/m}^3} = \frac{\text{m}^4}{\text{osmol} \cdot \text{s}}. \end{aligned} \tag{176}$$

The unit *osmol* is used in [3], but is actually not necessary since the number of ions for thin

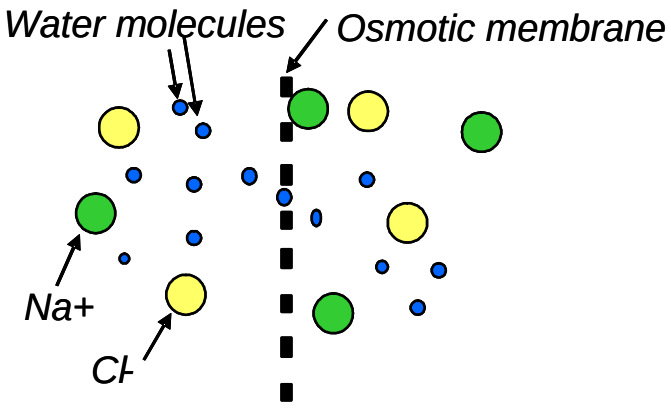

Figure 15: An osmotic membrane in a salt solution lets water molecules through, but not Chlorine and Sodium ions.

salt solutions will be twice the number of salt molecules, and therefore proportional to the salt concentration measured in kg/m^3.

We shall later return to the concepts of *diffusion* and *flux*. The transport of dissolved salt in a solution that is otherwise at rest, is mainly due to concentration differences:

$$F = -D\frac{\partial C}{\partial x}. \tag{177}$$

Here F is called *diffusive flux of* salt (in the x-direction), and D is called the diffusion coefficient.

Flux is in general the amount that passes through an imaginary surface per time and unit area. The unit of flux of salt is thus

$$[F] = \frac{\text{Amount}}{\text{Area} \times \text{Time}} = \frac{\text{osmol}}{\text{m}^2\text{s}}, \tag{178}$$

and since $[C]$ =osmol/m^3, then, from Eq.177,

$$[D] = \frac{\text{m}^2}{\text{s}}. \tag{179}$$

If a salt solution with concentration C is moving in the x-direction with velocity V, the expression for the *convective flux* is derived by considering Fig. 16. During one second the shaded volume passes through the surface with area A.

The expression for the flux is thus

$$F_{konv} = \frac{V \cdot 1s \cdot A \cdot C}{A \cdot 1s} = VC. \tag{180}$$

The total flux will be the sum of the contributions (this will be discussed in more depth later):

$$F = F_{diff} + F_{konv} = -D\frac{dC}{dx} + VC. \tag{181}$$

Chemical pumps are able to push the saline through the tissue (the body) by the consumption of chemical energy (that is about all that this author knows about it). The strength of a chemical pump is defined by its strength, N_0,

$$[N_0] = \frac{\text{Amount}}{\text{Wall area} \times \text{Time}} = \frac{\text{osmol}}{\text{m}^2\text{s}}. \tag{182}$$

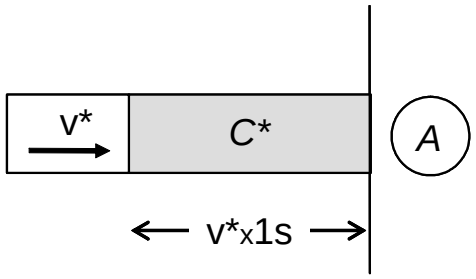

Figure 16: *Convective flux*: The solution with a salt concentration C^* is moving with speed v^* in the x-direction.

3.3.1 Formulation of the mathematical model

Figure 17 is a sketch of how we can imagine a one-dimensional mathematical model of the channel in the form of a straight tube of length, L, cross section, A, radius, c, and an active zone for the chemical pumps of length δ. We need to ask the biomedical and physical expertise to get an idea forthe size of the variables we have introduced. We shall return to the scaling below, but it is already here worth noting that the channels are *thin* (the diameter is only 0,001–0.01 times their length), so it must be sufficient to imagine a *one-dimensional* model.

As usual, we let non-scaled variables have an extra *, which is removed after the scaling. The water coming in through the wall (per unit area) due to osmosis at the location x^* will be

$$J = P\left[C^*(x^*) - C_0\right]. \tag{183}$$

The salt in the channel is transported both by diffusion and convection, and therefore

$$F^* = F_d + F_c = v^*C^* - D\frac{dC^*}{dx^*}. \tag{184}$$

The chemical pumps enter an amount of salt per time unit equal to $N_0(\delta c)$.

We have been asked to determine the so-called *emergent osmolarity*, Os^*, defined by $F^*(L) = v^*(L)Os^*$:

$$Os^* = \frac{F^*(L)}{v^*(L)} = \frac{v^*(L)C^*(L) - D\frac{dC^*}{dx^*}(L)}{v^*(L)}. \tag{185}$$

This is the equivalent salt concentration that would have the same outflow of salt, if the solution had speed v^* and there was no contribution from diffusion.

The basic principle of the modeling is the *preservation* or *conservation* of salt and water. This is the mathematical concept for the everyday expression that "what goes in must come out". Conservation of water for a part of the channel between x^* and $x^* + \Delta x^*$ can be illustrated as in 18. The terms of Q_{inn} and Q_{ut} (flow of water in and out per time unit) are simple:

$$\begin{aligned} Q_{inn} &= Av^*(x^*), \\ Q_{out} &= Av^{\acute{*}}(x^* + \Delta x^*) \end{aligned} \tag{186}$$

(Check this out by a similar argument to the one in Fig. 16). For Q_{os} it must be possible to write

$$Q_{os} = P\left(C^*(x^* + \tilde{x}^*) - C_0\right) \cdot (\Delta x^* \cdot c), \tag{187}$$

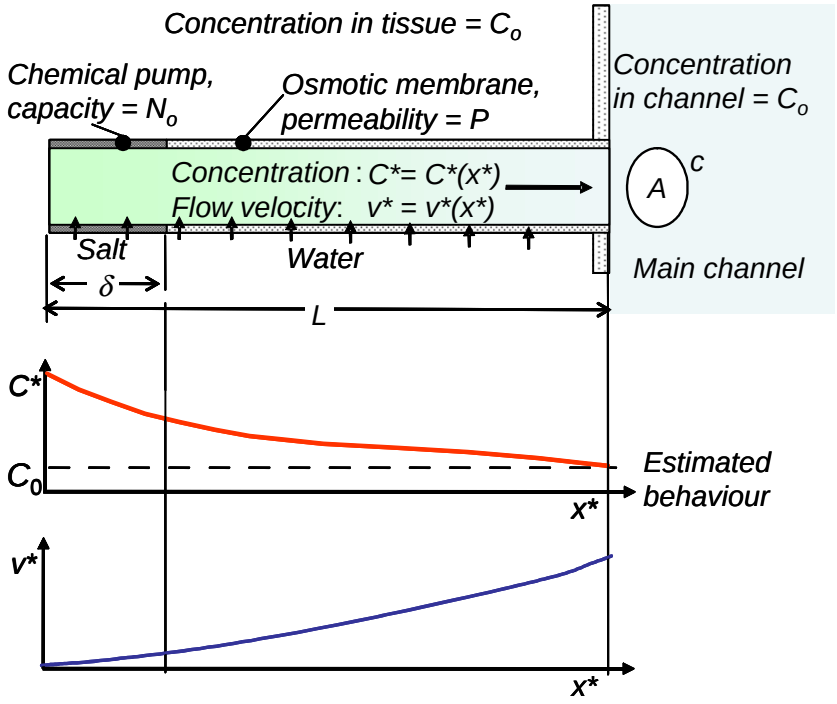

Figure 17: Sketch of one-dimensional mathematical model of the secondary channel.

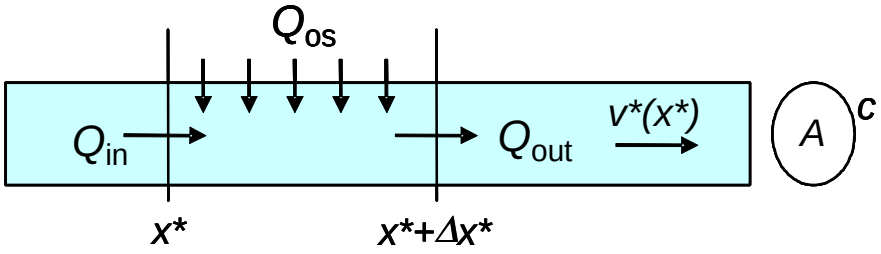

Figure 18: Illustration of the conservation of water for the section of the channel between x^* and $x^*+\Delta x^*$.

where $x^* \leq \tilde{x}^* \leq x^* + \Delta x^*$, and $\Delta x^* \cdot c$ is the wall area. Since $Q_{out} - Q_{in} = Q_{os}$, we have

$$Av^*(x^* + \Delta x^*) - Av^*(x^*) = P\left(C^*(x^* + \tilde{x}^*) - C_0\right) \cdot (\Delta x^* \cdot c)\,. \tag{188}$$

By dividing by Δx^* and letting $\Delta x^* \to 0$, we find the differential equation

$$\frac{dv^*}{dx^*} = \frac{Pc}{A}\left[C^*(x^*) - C_0\right]. \tag{189}$$

Conservation of the salt can be set up just as easily:

$$\begin{aligned} Q_{ut}^{salt} &= AF^*(x^* + \Delta x^*), \\ Q_{inn}^{salt} &= AF^*(x^*), \\ Q_{k.p.}^{salt} &= N^*(x^*) \cdot (c\Delta x^*)\,, \end{aligned} \tag{190}$$

where $N^*(x^*) = N_0$ when $x^* \leq \delta$, and equal to 0 otherwise. By letting $\Delta x^* \to 0$, we derive in a similar way

$$A\frac{dF^*}{dx^*} = \begin{cases} N_0 c, & x^* < \delta, \\ 0 & \delta \leq x^*. \end{cases} \tag{191}$$

This simple equation can be solved immediately. For $x^* < \delta$ we find, since $F^*(0) = 0$ (we assume that nothing enters through the end surface),

$$F^* = \frac{N_0 c}{A} x^*. \tag{192}$$

For the rest of the channel flux is constant. Since the flux has to be continuous at $x^* = \delta$ (think of what that means!),

$$F^* = \frac{N_0 c}{A}\delta. \tag{193}$$

This gives us now the following differential equation for C^*:

$$C^* v^* - D\frac{dC^*}{dx^*} = \begin{cases} \frac{N_0 c}{A} x^*, & x^* < \delta, \\ \frac{N_0 c}{A}\delta, & \delta \leq x^*. \end{cases} \tag{194}$$

Together with equation 189 we have got two non-linear, coupled differential equations for C^* and v^*. Before we try to solve them, we need to specify boundary conditions for the ends of the channel, as well as the continuity conditions (also called "matching" conditions) at $x^* = \delta$.

At $x^* = 0$ it is clear that

$$v^*(0) = 0,\ F^*(0) = 0. \tag{195}$$

(We consider the end of the channel to be closed). This implies, by means of equation 194, that

$$dC^*/dx^*\,(0) = 0.$$

At the outer end of the channel, it is reasonable to use

$$C^*(L) = C_0. \tag{196}$$

Finally, we consider $x^* = \delta$. It is clear that both F^*, v^* and C^* must be continuous (Think about it. Diffusion ensures that C^* does not develop discontinuities). Thus we may write

$$\begin{aligned} F^*(\delta+) &= F^*(\delta-), \\ v^*(\delta+) &= v^*(\delta-), \\ C^*(\delta+) &= C^*(\delta-). \end{aligned} \tag{197}$$

Quantity	Unit	Min. value	Typical value	Max. value
r	cm	10^{-6}	5×10^{-6}	10^{-4}
L	cm	4×10^{-4}	10^{-2}	2×10^{-2}
δ	cm	4×10^{-5}	10^{-3}	2×10^{-3}
D	cm^2/s	10^{-6}	10^{-5}	5×10^{-5}
N_0	$\mathrm{mOsm}/\mathrm{cm}^2\mathrm{s}$	10^{-10}	10^{-7}	10^{-5}
P	cm^4/s mOsm	10^{-6}	2×10^{-5}	2×10^{-4}
C_0	$\mathrm{mOsm}/\mathrm{cm}^3$	–	3×10^{-1}	–

Table 6: Overview of variables and parameters of the model. Here r is the radius of the channel, assumed to have a circular cross section.

Check from Eq. 194 that then also

$$\frac{dC^*}{dx^*}(\delta+) = \frac{dC^*}{dx^*}(\delta-). \tag{198}$$

We sum up the mathematical model:

Equations:

$$\frac{dv^*}{dx^*} = \frac{Pc}{A}\left(C^*(x^*) - C_0\right), \quad 0 \le x^* \le L, \tag{199}$$

$$C^*v^* - D\frac{dC^*}{dx^*} = \begin{cases} \frac{N_0c}{A}x^*, & x^* < \delta, \\ \frac{N_0c}{A}\delta & \delta \le x^*. \end{cases} \tag{200}$$

Boundary conditions:

$$v^*(0) = 0, \quad C^*(L) = 0, \tag{201}$$

$$v^*, C^* \text{ is continuous for } x^* = \delta. \tag{202}$$

Determine

$$Os^* = \frac{F^*(L)}{v^*(L)} = \frac{cN_0\delta/A}{v^*(L)}. \tag{203}$$

3.3.2 Scaling

Here we need first to obtain information about the values of variables parameters involved, and Table 6 has been copied from [3] (Table 8.2, p. 264).

We may use both L and δ as the length scale, but choose we follow [3] and use δ. The concentration scale is self-evident, namely C_0, while the velocity scale requires some creativity. Lin and Segel [3] propose to compute the velocity scale U by balancing the amount of salt produced per time unit with the amount leaving the channel by convection if the concentration is C_0. In other words,

$$cN_0\delta = C_0 \times (AU), \tag{204}$$

or

$$U = \frac{cN_0\delta}{C_0A}. \tag{205}$$

We may then set

$$x^* = \delta x, \quad C^* = C_0C, \quad v^* = Uv, \tag{206}$$

Dimensionless parameter	Min. value	Typical value	Max. value
ε	10^{-5}	2×10^{-2}	10^{2}
η	4×10^{-3}	75	10^{10}
λ	10	10	500

Table 7: The size range of the dimensionless parameters.

and leave to the reader to check that we can write the result as

$$\begin{aligned} \varepsilon\frac{dv}{dx} &= C-1, \;\; 0\le x\le\lambda, \\ Cv-\eta\frac{dC}{dx} &= \begin{cases} x, & 0\le x\le 1 \\ 1 & 1\le x\le\lambda \end{cases}, \end{aligned} \tag{207}$$

with conditions

$$v(0)=0, \;\; C(\lambda)=1, \tag{208}$$

$$v,\; C,\; dC/dx \text{ continuous at } x=1, \tag{209}$$

and the three dimensionless parameters:

$$\begin{aligned} \varepsilon &= \frac{N_0}{PC_0^2}, \\ \eta &= \frac{AC_0D}{N_0\delta^2 c}, \\ \lambda &= \frac{L}{\delta}. \end{aligned} \tag{210}$$

The dimensionless emergent osmolarity then becomes

$$Os = \frac{Os^*}{C_0} = \frac{1}{v(\lambda)}. \tag{211}$$

The range of magnitude of the dimensionless variables can now be estimated from Table 6, and the result is shown in Table 7.

3.3.3 Perturbation Analysis

Our equations do not look difficult at first glance, but they are nonlinear and coupled, and the initial conditions of C and v are given at the opposite ends of the channel.

Since ε will typically be small, it is natural to attempt a perturbation series, and Lin and Segel describes here how they first got stuck by inserting

$$\begin{aligned} C &= C_0+\varepsilon C_1+\varepsilon^2C_2+..., \\ v &= v_0+\varepsilon v_1+\varepsilon^2 v_2+... \end{aligned} \tag{212}$$

in equations 207 (Note that C_0 is here 0-th order solution for C and not the concentration scale). Check that we obtain $C_0=1$ to order ε , and

$$1v_0+\eta\frac{dC_0}{dx} = \begin{cases} x \\ 1 \end{cases}, \tag{213}$$

that is,

$$v_0 = \begin{cases} x & 0 \le x \le 1, \\ 1 & 1 \le x \le \lambda. \end{cases} \tag{214}$$

This looks good so far, but to order ε we derive

$$C_1(x) = \frac{dv_0}{dx} = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & 1 \le x \le \lambda, \end{cases} \tag{215}$$

in other words, C_1 is discontinuous. Also the perturbation solutions should be continuous!

It is therefore something fundamentally wrong with the assumptions we have made. If we look at the expressions for ε and η, N_0 occurs in the numerator in ε and in the denominator for η. In addition, it appears that N_0 can vary over more than 12 powers of 10. This means that if ε is small, then η tends to be large. Lin and Segel now try to introduce more stable parameter κ defined by

$$\eta = \frac{\left(\lambda^2/\kappa^2\right)}{\varepsilon}. \tag{216}$$

The essential point here is that $\eta \propto \varepsilon^{-1}$, whereas the chosen form simplifies the expressions. If we insert this, we get a set of modified equations,

$$C - 1 = \varepsilon \frac{dv}{dx}, \quad 0 \le x \le \lambda, \tag{217}$$

$$\varepsilon\kappa^2 Cv - \lambda^2 \frac{dC}{dx} = \varepsilon\kappa^2 \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 \le x \le \lambda \end{cases}, \tag{218}$$

$$v(0) = 0, \quad C(\lambda) = 1, \tag{219}$$

$$v,\ C,\ dC/dx \text{ continuous at } x = 1. \tag{220}$$

Moreover,

$$\varepsilon = \frac{N_0}{PC_0^2}, \quad \kappa = \left(\frac{\lambda^2}{\eta\nu}\right)^{1/2} = \left(\frac{cPC_0L^2}{AD}\right)^{1/2}, \ \lambda = \frac{L}{\delta}. \tag{221}$$

Before we start a new perturbation analysis we can test the problem with some numerical experiments. The equations are a first-order system, but to solve them in the standard way, we must start $C(x)$ and $v(x)$ in the same point. If we choose to start at 0, then we need to vary $C(0)$ so that we really hit $C(\lambda) = 1$. This is called solving by a "shooting method". The search for the starting value can be build into the program, but here it is just as easy to use "trial and error". Matlab^{TM}-code needed to solve the equations is simple:

```
x0 =[0, 1.0355]; tspan = [0, 10];
[t,x]=ode45('nyre',tspan,x0);
subplot(2,1,1);plot(t,x(:,2));
subplot(2,1,2);plot(t,x(:,1));
```

The function that defines the equations:

```
function xdot=nyre(t,x)
lambda = 10.;
kappa = 1;
eps = .5;
```

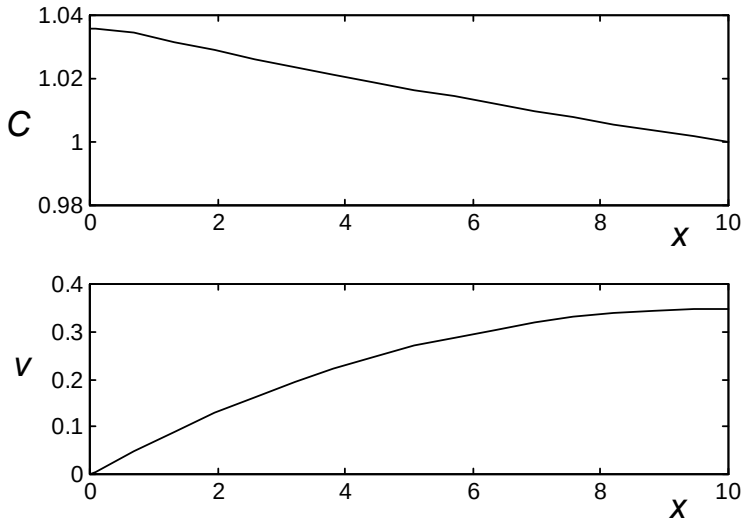

Figure 19: Solution obtained with the "shooting method". Parameter values: $\lambda = 10, \quad \kappa = 1, \quad \varepsilon = .5$. In order to end in 1 for $x = 10$, $C\,(0)$ must be about 1.0355.

```
A= eps*kappa^2/lambda^2;
xdot(1)=( x(2)-1 )/eps;
xdot(2)=A*x(1)*x(2)-A*min(t,1);
```

A sample result is shown in Figure 19. Let us now look at the modified equations, and try a perturbation analysis as given in Eq. 212. To order ε^0 we obtain as before $C_0 = 1$, but now the equations to order ε are more interesting:

$$C_1 = \frac{dv_0}{dx}, \tag{222}$$

$$\kappa^2 C_0 v_0 - \lambda^2 \frac{dC_1}{dx} = \kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{223}$$

that is,

$$\kappa^2 v_0 - \lambda^2 v_0'' = \kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix} \tag{224}$$

for v_0. This equation must be solved for both the right side and put together so that the boundary condition at 0 and continuity conditions in $x = 1$ holds. We leave it to readers to check the following solution for $0 \le x \le 1$:

$$v_0 = x - K_1 \sinh\left(\frac{\kappa}{\lambda} x\right), \tag{225}$$

$$C_1 = v_0' = 1 - K_1 \frac{\kappa}{\lambda} \cosh\left(\frac{\kappa}{\lambda} x\right). \tag{226}$$

The solution for $1 \le x \le \lambda$:

$$v_0 = 1 - K_2 \cosh(\frac{\kappa}{\lambda} x - \kappa), \tag{227}$$

$$C_1 = v_0' = -K_2 \frac{\kappa}{\lambda} \sinh\left(\frac{\kappa}{\lambda} x - \kappa\right) \tag{228}$$

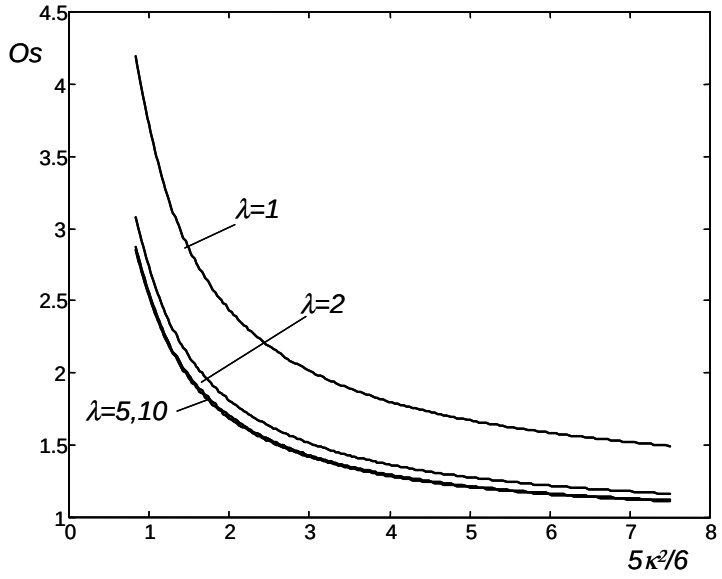

Figure 20: Dimensionless osmolarity to first order in Eq. \ref{os}.

(note that $C_1(\lambda) = 0$!).

The constants K_1 and K_2 are determined by the continuity conditions at $x = 1$:

$$K_1 = \frac{\lambda}{\kappa}\frac{\cosh(\kappa/\lambda - \kappa)}{\cosh(\kappa)}, \tag{229}$$

$$K_2 = \frac{\lambda}{\kappa}\frac{\sinh(\kappa/\lambda)}{\cosh(\kappa)}. \tag{230}$$

The dimensionless osmolarity is approximately

$$Os = \frac{1}{v(\lambda)} \approx \frac{1}{v_0(\lambda)} = \frac{1}{1 - K_2}, \tag{231}$$

and plotted for some values of λ on Fig. 20.

If we assume that $\kappa/\lambda < 1$, we may write

$$Os \approx \frac{1}{1 - K_2} \approx \frac{1}{1 - \frac{1}{\cosh(\kappa)}} = \frac{\cosh(\kappa)}{\cosh(\kappa) - 1}, \tag{232}$$

and if in addition $\kappa < 1$,

$$Os \approx \frac{\cosh(\kappa)}{\cosh(\kappa) - 1} \approx \frac{1 + \kappa^2/2 + ...}{\kappa^2/2 + ...} \approx \frac{2}{\kappa^2} + 1. \tag{233}$$

3.3.4 Epilogue

Lin & Segel [3] conducted this analysis in the seventies. Today we would have started with carrying out some numerical experiments. But since the problem has three dimensionless parameters, it is difficult to obtain a full understanding only by doing this. It is interesting that the straightforward

"naive" perturbation expansion breaks down, and that the modified perturbation solution seems to give a very reasonable answer already to leading order (details are left to the reader to check out).

To leading order the emergent osmolarity is only dependent on λ and κ. The perturbation expansion therefore gives an analytical insight that is not so easily obtained by numerical experiments alone.

The dimensionless parameters that we end up with often express important properties of the model we have created. This is the case also in this example, where the parameter κ can be interpreted as follows:

$$\frac{\kappa^2}{2} = \frac{PcC_0L^2}{AD2} = \frac{PcC_0L^2}{AD2}\frac{\bar{C}-C_0}{\bar{C}-C_0} = \frac{\left(cLP(\bar{C}-C_0)\right)C_0\frac{1}{A}}{D\left(\bar{C}-C_0\right)/(L/2)}. \tag{234}$$

($\bar{C}$ is the mean concentration in the channel). The water entering by osmosis is approximately equal to $cLP(\bar{C}-C_0)$, and consequently the convective flux of salt out of the channel is

$$F_{conv.} \approx PcL(\bar{C}-C_0)\times C_0 \times \frac{1}{A}. \tag{235}$$

Similarly, we can write

$$F_{diff} \approx -D\overline{\left(\frac{dC^*}{dx^*}\right)} \approx D\frac{\bar{C}-C_0}{L/2}. \tag{236}$$

Thus,

$$\frac{\kappa^2}{2} \approx \frac{F_{conv.}}{F_{diff.}}, \tag{237}$$

and

$$Os = \frac{Os^*}{C_0} = \frac{F^*(L)}{v^*(L)C_0} = \frac{F_{conv.}+F_{diff}}{F_{conv.}} = 1 + \frac{F_{diff}}{F_{conv.}} \approx 1 + \frac{2}{\kappa^2}, \tag{238}$$

which we also found above.

References

[1] Barenblatt, G.I.: *Scaling, Self-similarity, and Intermediate Asymptotics*, Cambridge Texts in Applied Mathematics, 1996.

[2] Brigham, P.W.: *Dimensional Analysis,* Yale University Press, 1922.

[3] Lin, C.C. and L.A. Segel: *Mathematics Applied to Deterministic Problems in the Natural Sciences*, SIAM Classics in Applied Mathematics, 1988.

[4] Munson, B.R., D.F. Young and T.H. Okiishi: *Fundamentals of Fluid Mechanics,* Wiley 1990.

[5] Sedov, L.I.: *Similarity and Dimensional Methods in Mechanics*, Acad. Press, 1959.

[6] `http://www.cut-the-knot.org/pythagoras/index.shtml`

[7] Taylor, E.S.: *Dimensional Analysis for Engineers,* Clarendon Press, 1974.

4 SELECTED EXERCISES

Hint: Use *Wikipedia* or *Internet* for sorting out basic physical concepts.

4.1 Dimensional Analysis

Exercise 1

State the SI-units for the following physical quantities: (*i*) Acceleration, (*ii*) Mass density, (*iii*) Electrical power, (*iv*) Air pressure, (*v*) Specific heat capacity, (*vi*) Heat conduction coefficient.

Exercise 2

Mechanical stress has the same unit as pressure (Force per unit area). For a *Newtonian fluid* (like water and air) flowing in the x-direction the so-called *shear stress* on a plane parallel to the xy-plane is given by

$$\tau = \mu \frac{\partial u\left(x, y, z\right)}{\partial y}, \tag{239}$$

where u is the velocity in the x-direction at (x, y, z). What is the unit for the constant μ, called the *dynamic viscosity*?

Exercise 3

Which combinations of *core variables* from the set $\{R_1, \cdots, R_6\}$ may be used if the dimensional matrix is

	R_1	R_2	R_3	R_4	R_5	R_6
F_1	1	1	-1	0	2	2
F_2	-2	-1	1	0	-3	-2
F_3	0	1	0	1	0	2

.

Exercise 4

An open cylindrical tank with diameter, D, is filled to height, h, with a fluid of density, ρ. The bottom has thickness, d, and an elasticity module, E (E is measured in Pascal, like stress). Because of the weight of the fluid, the bottom will sink somewhat, most at the center (No sinkage at the rims). Show that the sinking (distance, δ) in the centre of the bottom may be expressed as

$$\frac{\delta}{D} = \Phi\left(\frac{h}{D}, \frac{d}{D}, \frac{E}{Dg\rho}\right), \tag{240}$$

where g is the acceleration of gravity.

Exercise 5

A skydiver in *free fall* with speed U experiences a *drag* (friction force) from the surrounding air. The drag may be written as

$$F_d = \frac{1}{2}\rho_{\text{air}} A U^2 \phi\left(\frac{U\sqrt{A}}{\nu}\right), \tag{241}$$

where ρ_{air} is the density of air, A is the cross-sectional area of the skydiver, and ν the kinematic viscosity of the air.

(a) Show how this expression for F_d may be found by dimensional analysis (*Hint*: Use, if necessary, the formula in Eq. 241 first to determine the units of the involved parameters).

(b) After a while the free fall jumper will be falling with constant speed. Find an expression for this speed if we assume that $\phi(x) = 1$.

(*Hint*: The force of gravity, pulling the skydiver downwards, is $F_g = mg$, where m is the skydiver's mass and g the acceleration of gravity. Use that F_g is equal to F_d when the speed is constant).

(c) Estimate the free fall speed in km/hour if we assume that $\phi\,(x) = 1$.

Exercise 6

An industrial tank holding a chemical liquid has a hole near the bottom. The chemical is flowing through the hole at an amount Q, measured in m^3/s. It is reasonable to assume that Q depends on the diameter of the hole and the pressure difference Δp in the fluid between the inner and outer sides of the hole. In addition, we expect that the flow is governed by the fluid's density ρ and dynamic viscosity μ. Use dimensional analysis to show that the expression for Q under these assumptions may be written

$$Q = \frac{d^2 \Delta p^{1/2}}{\rho^{1/2}} \phi \left(\frac{d \rho^{1/2} \Delta p^{1/2}}{\mu} \right), \tag{242}$$

where ϕ is an unknown function of only one variable.

Exercise 7

The force (F) on an aircraft propeller depends on its diameter, d, the speed of the airplane, U, the density of the air, ρ, the number of rotations per second, ω, and the viscosity of the air, μ.

Show how dimensional analysis is used to find the formula

$$F = \rho U^2 d^2 \phi \left(\frac{\omega d}{U}, \frac{U d}{\mu/\rho} \right), \tag{243}$$

where ϕ is an unknown function in two variables (Hint: Use, if necessary, Eq. 243 to find the units for the variables).

Exercise 8

By measuring the pressure drop p in a tube vs. the time t it took to fill a cup with volume V, *Bose*, *Bose* and *Ruert* around 1910 found the relations on Fig. 21 (left) for water, chloroform, bromoform and mercury. Show, by introducing dimensionless variables (using the density ρ and viscosity μ, $[\mu] = \mathrm{kg\ s^{-1} m^{-1}}$), that it exists one common relation covering all the cases. That is, find the variables along the axes in *von Kárman's* alternative presentation of the same data, as shown in the figure to the right.

Exercise 9

We consider an elastic rubber band which may be stretched many times its original length l_0. The rubber band has a "density" ρ which we measure in mass per unit length, that is, kg/m. How does the density ρ vary when we stretch the band to a length l from its original length l_0 and density ρ_0? After stretching the band more that twice its original length, we pluck the band like a guitar string. This experiment shows, somewhat unexpected, that the *pitch* (=frequency ω) remains almost constant when we vary the length (*try it*!). However, when stretching the band up towards its breaking limit, the frequency increases somewhat. The force F required for stretching the band to length l is proportional to $l - l_0$ over most of the range, that is, $F = F_0 \frac{l - l_0}{l_0}$, where F_0 is a constant. Use dimensional analysis to explain the behaviour of the frequency. (*Hint*: Assume

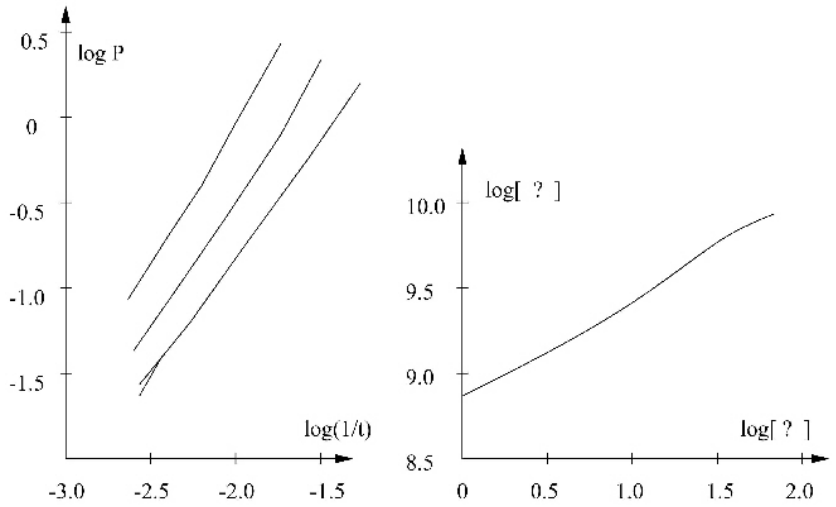

Figure 21: Presentation of the data in the original paper (left) and vonKarman's revised graph after applying dimensional analysis (right).

first that $\omega = f(l, \rho, F)$, apply dimensional analysis, and then introduce the expression of the density as a function of l, l_0 and ρ_0).

Exercise 10

In forest assessment one wants to estimate the volume V (also called the *cubic content*) of a tree by measuring its height (h) and diameter (d) at the root. A test example in *Minitab* suggests the following formula (based on multilinear regression) for American cherry trees:

$$V^{1/3} = \beta_0 + \beta_1 d + \beta_2 h + \beta_3 d^2. \tag{244}$$

Here $\{\beta_i\}$ are regression coefficients calculated from a set of calibration data.

(a) Americans use *foot* and most of the rest of the world *meter* as the basic length unit. Is it possible to use the same values for $\{\beta_i\}$ in both cases?

(b) Show that dimensional analysis, based on the variables v, d, and h, instead recommends applying a relationship of the form

$$\pi_1 = \phi(\pi_2). \tag{245}$$

Find π_1 and π_2, and give examples of what the function ϕ could be for some "idealized trees", e.g. cylinders and cones.

(For students who know *Minitab*, it might be interesting to check which of the models are best: The model in Eq. 244, or a regression model based on 245. The calibration data used in Minitab 14 are stored in the MTW file `trees.mtw`)

Exercise 11

The necessary force (F) to keep a ship at a constant speed (U) depends on its shape; primarily the length (L), width (W), and its depth into the water (D). In addition, the water density, ρ, the viscosity, ν, and the acceleration of gravity, g, play a part.

Use dimensional analysis to find an expression for the force which includes the two most famous

dimensionless numbers in ship design:

$$\textit{Froude number}:\ Fr = U/\sqrt{Lg}, \tag{246}$$

$$\textit{Reynolds number}:\ Re = LU/\nu. \tag{247}$$

Ideally, a scale model[1)] of the ship should be tested experimentally in water by keeping the dimensionless numbers for the model equal to those of the original ship. Is this really possible?

(*Hints*: $[F] = \mathrm{kgm/s^2}$, $[\rho] = \mathrm{kg/m^3}$, $[\nu] = \mathrm{m^2/s}$, $[g] = \mathrm{m/s^2}$).

[1)] A *scale model* is a model of the ship with the same geometric shape, but with a smaller size (Say, L =1m for the model, compared to 200m for the original ship).

4.2 Scaling and Regular Perturbation

Exercise 1

State what it means to scale (i) *a physical quantity*, (ii) *an equation.*

Exercise 2

The following expressions have been proposed as the time scale for the function $u^*(t^*) = A\cos(2\pi f_0 t^*)$:

$$\begin{aligned} &T = 1/f_0, \\ &T = 1/(2\pi f_0), \\ &T = 1/(\pi f_0), \\ &T = 500\pi f_0^{-1}. \end{aligned} \tag{248}$$

May all these be used as the time scale?

Exercise 3

A common mathematical model for the size of a population $y^*(t^*)$ as a function of time t^* is described by the *logistic equation*

$$\frac{dy^*}{dt^*} = ry^*\left(1 - \frac{y^*}{K}\right). \tag{249}$$

Here r is called the *growth rate* and K the *sustainable capacity.*

(a) Which scale is suitable for y^*?

(b) Determine a time scale when $y^* \ll K$.

(c) Introduce these scales into the equation so that it becomes dimensionless (The equation can easily be solved by inserting $y = 1/u$ and solving for u).

Exercise 4

In many dynamic systems one talks about *time constants.* For an exponential function, $u(t) = A\exp(-at)$, the *time constant* is defined as follows: First draw the tangent to $u(t)$ at t_0. This tangent is crossing the x-axis at t_1, and the time constant is defined $T = |t_1 - t_0|$. Show that this definition also follows from the *rule of thumb*,

$$T = \frac{\max|u(t)|}{\max|du(t)/dt|}. \tag{250}$$

Exercise 5: Case B in the discussion of the falling sphere in a fluid led to the equation

$$2\ddot{x} + \varepsilon\dot{x} = 1,\ x(0) = 0,\ \dot{x}(0) = 0,\ 0 < \varepsilon \ll 1. \tag{251}$$

This equation has the exact solution

$$x_{\text{sol}}(t) = \frac{2}{\varepsilon^2}\left(e^{-\varepsilon t/2} - 1\right) + \frac{t}{\varepsilon}. \tag{252}$$

(a) Determine x_0, x_1 and x_2 in the regular perturbation expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots, \tag{253}$$

and show that it agrees with the start of the power series development in ε of the exact solution.

(b) An approximate solution $x_a(t, \varepsilon)$ is a *uniform approximation* to the exact solution, x_{sol}, on the interval $[0, 1]$ if

$$\lim_{\varepsilon \to 0}\left(\max_{t \in [0,1]} |x_a(t, \varepsilon) - x_{\text{sol}}(t)|\right) = 0. \tag{254}$$

Does this apply to $x_a(x, \varepsilon) = x_0(t) + \varepsilon x_1(t)$? What if we replace $[0, 1]$ with$[0, \infty)$?

Exercise 6

Consider the problem

$$y''(t) + \varepsilon y'(t) + 1 = 0, \tag{255}$$

$$y(0) = 0\ ,\ y'(0) = 0,\quad 0 < \varepsilon \ll 1. \tag{256}$$

Determine the start of the perturbation expansion $y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)$ to the solution for $t \geq 0$. Compare to the exact solution. (*Hint*: The general solution of Eq. 255 has the general form $y(t) = A + Be^{-\varepsilon t} - t/\varepsilon$)

Exercise 7

This problem is somewhat similar to the sphere falling in a fluid (the scaling model problem without gravity), but in this case the friction is more realistic and nonlinear. The equation may be written

$$m\frac{dv^*}{dt^*} = -av^* + bv^{*2},\quad v^*(0) = V_0. \tag{257}$$

and

$$v^*(0) = V_0. \tag{258}$$

We have been told that $a, b > 0$, and also that $bV_0 \ll a$.

(a) Find the (obvious) scale for v^* and then the scale for time, T, from the simplified equation, $m\frac{dv^*}{dt^*} = -av^*$, by the *rule of thumb*

$$T = \frac{\max|v^*(t)|}{\max|dv^*(t)/dt^*|}. \tag{259}$$

Show that this scaling leads to the equation

$$\frac{dv}{dt} = -v + \varepsilon v^2, \tag{260}$$

$$v(0) = 1,\ \varepsilon \ll 1. \tag{261}$$

(b) Determine v_0 and v_1 in the series expansion $v(t) = v_0(t) + \varepsilon v_1(t) + \cdots$. Is this result reasonable for all $t > 0$ when the general solution of $\dot{y} + y - \varepsilon y^2 = 0$ is

$$y(t) = \frac{e^{-t}}{C + \varepsilon e^{-t}}? \tag{262}$$

Exercise 8

During the modeling of the sprinters, we derived the equation

$$M\frac{du^*}{dt^*} = Mp^*(t^*) - M\frac{u^*}{\tau} - F_{\text{air}}, \tag{263}$$

where M is the runner's mass, u^* the velocity, p^* a "performance variable", τ a time constant, and F_{air} the air resistance.

(a) Explain why the term for air resistance, found from dimensional analysis, ought to be stated as

$$F_{\text{air}} = \tfrac{1}{2}\rho_{\text{air}} C_D(Re)\, A(u^* - W)\,|u^* - W|\,. \tag{264}$$

(Here ρ_{air} is the air density, Re the Reynolds number and A the runner's cross-sectional area).

After scaling,

$$p^* = Pp, \tag{265}$$
$$t^* = \tau t, \tag{266}$$
$$u^* = (P\tau)\,u, \tag{267}$$

and without wind, the equation becomes

$$\dot{u}(t) + u(t) + \varepsilon u(t)^2 = p(t), \tag{268}$$
$$u(0) = 0, \tag{269}$$

where

$$\varepsilon = \tfrac{1}{2}\rho_l C_D(Re)\,\tau^2 P\tfrac{A}{M}. \tag{270}$$

(b) Estimate ε for Usain Bolt and Florence Griffith-Joyner when we assume here and below that $\rho_{\text{air}} = 1.2\text{kg/m}^3$, $C_D(Re) \equiv 1$, $P = 10\text{m/s}^2$ and $\tau = 1\text{s}$.

(c) Verify that the solution of Eq. 268 with $p(t) \equiv 1$ will be

$$u(t) = 1 - e^{-t} + \varepsilon[-1 + 2te^{-t} + e^{-2t}] + \mathcal{O}\left(\varepsilon^2\right). \tag{271}$$

(d) We scale the wind speed in the same way as u^*, so that

$$u^* - W = (P\tau)\,(u - \delta)\,. \tag{272}$$

Show that the dimensionless maximum speed $u_{\max}$ (when $u_{\max} > \delta$) is given by

$$u_{\max} + \varepsilon(u_{\max} - \delta)^2 = 1. \tag{273}$$

(e) Show that in order to determine the more exact time spent on the running, T_a, on the basis of this model it will be necessary to solve the equation

$$\frac{L}{P\tau^2} = \int_0^{T_a/\tau} u\,(t)\,dt, \tag{274}$$

where $L = 100$m.

(f) The discussion in the last few paragraphs of the course note is rough, since it is assumed that the runner holds a maximum speed of during the whole distance. The advantage of the tailwind is therefore estimated too large. From the information about her time under controlled conditions, it is possible derive the size of ε, provided that the model holds.

So what is the conclusion of this study? *Tailwind* or *doping*?

Matlab code for those interested (This code may need some modifications for Octave):

```
% SCRIPT
global EPS DELTA
% Parameters:
P = 10;    % Maximum performance factor [m/s^2]
tau = 1.0; % Relaxation time [s]
M = 60;    % Body mass [kg]
A = 0.40;  % Cross sectional area [m^2]
rho = 1.2; % Density [kg/m^3]
W = 0 ;    % Wind speed [m/s]
Cd = 1;    % Drag coefficient
%
EPS = 0.5*rho*Cd*tau^2*P*A/M
DELTA = W/(P*tau)
%
Treal = 0:0.1:12; % NB! Time in seconds
tspan = Treal/tau;
% ODE solver
[t,y] = ode45(@FJfunc,tspan,[0 0]');
T = t*tau;        % Real time
L = P*tau^2*y;    % Real distance
plot(T,L); xlabel('Time [s]') ; ylabel('Distance [m]');
legend('Position','Velocity')
grid

function dydt = FJfunc(t,y)
global EPS DELTA
dydt    = [0 0]';
dydt(1) = y(2);
dydt(2) = 1-y(2)-EPS*(y(2)-DELTA)*abs(y(2)-DELTA);
```

TMA 4195
Mathematical Modeling

Supplementary Notes, Part 2:

Dynamic Population Models

Autumn 2011

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Contents

1 INTRODUCTION

This note is based on a Mathematical Modelling seminar in 1995 dealing with population models. The note is rather concise and assumes that the reader already has some knowledge in analysis of non-linear systems of differential equations. At the end of the note, you can find some of the references that the material is collected from, and an appendix which shows how to make a simple MATLAB program for examining some of the treated systems.

2 THE LOGISTIC EQUATION

The *logistic equation* is a non-linear first order differential equation that can be written

$$\frac{dN^*}{dt^*}\frac{1}{N^*} = r\left(1 - \frac{N^*}{K}\right). \tag{1}$$

The equation describes the number of individuals N^* in a single population as a function of time t^*. The constant K is called the sustainable level or capacity, while r is a growth rate. If $N^* \ll K$, the solution is approximately exponential $N^* = N_0 \exp(rt^*)$, with natural time scale $1/r$. The capacity K, which is also a stationary solution, gives us a scale for N^*. This leads to the scaled equation

$$\frac{dN}{dt} = N - N^2. \tag{2}$$

The most natural is to solve the equation by separation, but the simplest is to introduce $U = 1/N$, which gives the linear equation $\dot{U} + U = 1$, with general solution

$$U(t) = 1 + Ae^{-t}. \tag{3}$$

Thus,

$$N(t) = (1 + Ae^{-t})^{-1}. \tag{4}$$

Depending on the sign of A, the solution may be expressed as

$$s(t) = \frac{1}{1 + e^{-(t-t_0)}} \tag{5}$$

for $A > 0$, and

$$s(t) = \frac{1}{1 - e^{-(t-t_0)}} \tag{6}$$

for $A < 0$.

We see that *all* solutions in the interval $\langle 0, 1\rangle$ are expressible by the single function

$$s(t) = \frac{1}{1 + e^{-t}}, \tag{7}$$

which is called the logistic curve, or the *sigmoid*. Solutions in the interval $\langle 1, \infty\rangle$ evolve according to the single function $1/(1 - e^{-t})$, as shown in figure 1.

We immediately see that $N \equiv 1$ is a stable equilibrium solution, while $N \equiv 0$ is unstable (*Physically*, $N = 0$ is of course stable!). One feature of the sigmoid is that no matter how much time it has taken a population to reach a certain level, e.g. $K/10$, it will only take time $\mathcal{O}(1/r)$

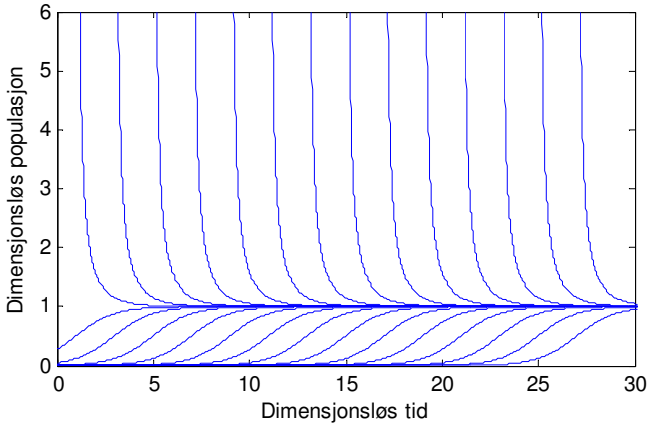

Figure 1: Solutions of the scaled and dimensionless logistic equation.

to reach saturation. We also note that no matter how high the starting level is, we will reach equilibrium in time $\mathcal{O}(1/r)$. Thus, changing K significantly has dramatic consequences.

It is useful to apply the trick of introducing $U = 1/N$ even when r and K vary with time. Let us change the notation above for a moment and write $K^*(t^*) = K\kappa(rt^*)$, $r^* = r\rho(rt^*)$. After scaling, we get the equation

$$\frac{dN}{dt} = \rho(t)N\left(1 - N/\kappa(t)\right), \tag{8}$$

or

$$\frac{dU}{dt} + \rho(t)U = \rho(t)/\kappa(t), \tag{9}$$

which is a linear first order equation. As a simple example, consider $\rho(t) = 1$ and $1/\kappa(t) = 1 + a\sin(\omega t)$. Then, after all transients have died out, the solution is

$$N(t) = \left\{1 + \frac{a}{\sqrt{1+\omega^2}}\sin\left(\omega t - \phi\right)\right\}^{-1}, \quad \phi = \arcsin\left(\frac{\omega}{\sqrt{1+\omega^2}}\right). \tag{10}$$

The behaviour when $\omega \to 0$ and $\omega \to \infty$ is as expected.

2.1 Hunting and Catching

It is easy to incorporate hunting or catching into the logistic model, e.g. the fish stock in a certain part of the ocean. The model starts from the logistic equation and additionally assumes a level of fishing per unit of time which is proportional to the fish stock and the number of boats u^* participating,

$$\frac{dN^*}{dt^*} = rN^*\left(1 - \frac{N^*}{K}\right) - \alpha N^* u^*. \tag{11}$$

By a similar scaling as above, the equation becomes

$$\frac{dN}{dt} = N - N^2 - \mu N, \quad \mu = \alpha u^*/r. \tag{12}$$

The equilibrium solutions become $N_1 = 0$, $N_2 = 1 - \mu$. Standard linear stability analysis gives that the solution $N_1 = 0$ is stable for $\mu > 1$ and unstable for $\mu < 1$, while $N_2 = 1 - \mu$ is stable when $\mu < 1$. Exploring the case $\mu = 1$ is left to the reader.

A steady withdrawal per time unit can be expressed as

$$\frac{dQ}{dt^*} = \alpha N_2 u^* = \alpha u^* K(1-\mu) = rK\mu(1-\mu), \tag{13}$$

which reaches a maximum for $\mu = 1/2$, i.e. $u^* = r/(2\alpha)$ and $N_2^* = K/2$. Hence, optimal resource management is attained when the stock is kept at a level with maximal growth, i.e. at half the maximal stock level. This is a well known "law" in resource management.

2.2 Delayed reaction

Delay in a logistic model enters naturally in systems where the growth is dependent on the conditions some time back. The equation is

$$\frac{dN^*}{dt^*}\frac{1}{N^*} = r\left(1 - \frac{N^*(t^*-\tau^*)}{K}\right) \tag{14}$$

which becomes, after scaling in the same way as before,

$$\frac{dN(t)}{dt} = N(t)\left(1 - N(t-\tau)\right). \tag{15}$$

In May 1973 the Norwegian Nobel prize winner in economics, Ragnar Frisch, was the first to start applying such models. In general, it is not possible to solve the equation analytically, but we still have the equilibrium equation $N(t) \equiv 1$. Linear stability analysis is about introducing $N(t) = 1 + a(t)$ and neglecting terms that are $O(a^2)$. This gives us the prototypical delay equation

$$\dot{a}(t) = -a(t-\tau). \tag{16}$$

The equation is linear, so the sum of two solutions is still a solution. Thus, it is reasonable to study the behaviour of Fourier components, and by inserting a component of the form $a(t) = a_0 e^{zt}$, $z = x + iy$, we get a complex equation for z,

$$z = -e^{-z\tau}, \tag{17}$$

or the following pair for (x, y):

$$\begin{aligned} x &= -e^{-x\tau}\cos(y\tau), \\ y &= e^{-x\tau}\sin(y\tau) \end{aligned} \tag{18}$$

Since $e^{-zt} = e^{-xt}e^{-iyt}$, the stability properties are determined by the value of x. It is easy to show that if $0 \le \tau < 1/e$, the solution is real with $-e < x < -1$. In other words, $N(t) = 1$ is stable. In the interval $1/e < \tau \le \pi/2$, the dominant solution will still have negative real part, but it will now be complex. This gives damped oscillations, which in the limit with small amplitude is exponentially damped. When $\tau = \pi/2$, the linearized equation has a periodic solution, and for larger τ there will be solutions where $x > 0$. Then, $N(t) = 1$ is no longer stable.

For further studies of delay equations, see the literature, e.g. the book by Hairer, Nørsett and Wanner (1993). It has been said that this type of equations explains the dramatic population cycle of lemmings (see figure 2). Note that there are freely available program packages on the Internet for solving delay equations in MATLAB.

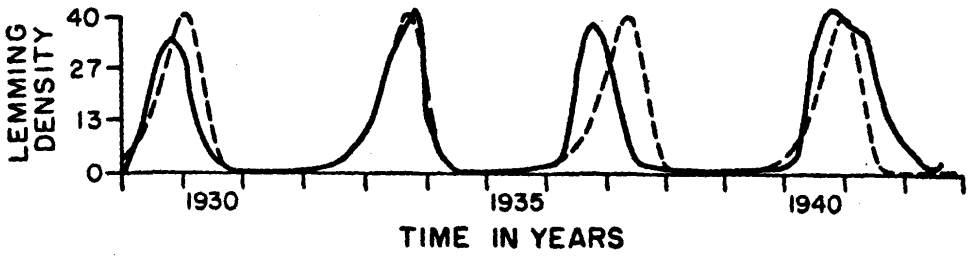

FIG. 2. Shelford's (1943) data on the lemming population in the Churchill area in Canada (expressed as numbers of individuals per hectare), compared with a naive theoretical curve (dashed line) obtained from the simple time delayed logistic equation (3); the time delay T is taken to be a little under one year ($T = 0.72$ yr.).

Figure 2: Lemming population and the delay logistic equation.

3 THE GROWTH OF THE EARTH POPULATION

In laboratory studies of closed bacterial cultures, it has proven hard to find a behaviour following the logistic model. Instead, the population eventually tends to zero because of *self-poisoning*. It's not improbable that something similar will hold for humanity. The poisoning may be caused by PCBs, long-lived radioactive isotopes, or hormone copycats affecting fertility. This has already occurred for our neighbours in the Arctic.

In a world where social benefits are more equally distributed, it would be natural that the amount of pollution released per unit time is proportional to the number of individuals. The total amount of pollution at time t^* is then proportional to

$$\int_{-\infty}^{t^*} N^*(s^*)\, ds^* \tag{19}$$

if the pollution does not brake down with time. In addition, we assume that the negative effect on the growth rate is proportional to the amount of pollution. This gives us the following modified model:

$$\frac{1}{N^*(t^*)}\frac{dN^*(t^*)}{dt^*} = r\left(1 - \frac{N^*(t^*)}{N_m}\right) - c\int_{-\infty}^{t^*} N^*(s^*)\, ds^*,\ c > 0. \tag{20}$$

(Note: This is not a new and original model, but the exact reference to the literature is missing).

Is it possible that a population following this equation would survive in the long term? Assume that $\lim_{t^*\to\infty} N^*(t^*) = a > 0$, and that the population development starts at $N_s < N_m$. The solution will stay below N_m and thus,

$$\frac{dN^*(t^*)}{dt^*} < N_m\left(r - c\int_{-\infty}^{t^*} N^*(s^*)\, ds^*\right). \tag{21}$$

Since $\lim_{t^*\to\infty} N^*(t^*) = a$, the right-hand side will tend to $-\infty$ when $t^* \to \infty$, which is completely incompatible with $N^*(t^*) \to a$.

A simple alternative model which doesn't have as dramatic consequences, is

$$\frac{1}{N^*(t^*)}\frac{dN^*(t^*)}{dt^*} = r\left(1 - \frac{N^*(t^*)}{N_m}\right) - c\int_{-\infty}^{t^*} e^{-(t^*-s^*)/\tau} N^*(s^*)\, ds^*. \tag{22}$$

Here, the pollution (or the effect on the growth rate) is broken down with time constant τ, and it seems that a certain population level can survive (Details are left for the reader).

Let us go back to equation 20 and scale it in the usual way. This results in

$$\frac{1}{N}\frac{dN}{dt} = 1 - N - \alpha \int_{-\infty}^{t} N(s)\, ds,\ \alpha = \frac{cN_m}{r^2}. \tag{23}$$

The equation reduces to a 2nd order equation by introducing $C(t) = \int_{-\infty}^{t} N(s)\, ds$, but it can probably not be solved analytically. If α is large, and we start with a small population, we can expect that N never reaches 1. Then, if we neglect the second term on the right-hand side, we obtain an equation which gives us an upper limit for the population developing according to the complete equation,

$$\frac{1}{N_u}\frac{dN_u}{dt} = 1 - \alpha \int_{-\infty}^{t} N_u(s)\, ds. \tag{24}$$

We introduce $P(t) = \alpha \int_{-\infty}^{t} N_u(s)\, ds$, which leads to the equation

$$P'' = P'(1 - P). \tag{25}$$

The equation can be integrated once,

$$P'(t) = P - \frac{P^2}{2} + A, \tag{26}$$

where A is a constant.

If we restrict ourselves to the situation where both $P(t)$ and $P'(t)$ tend to 0 when $t \to -\infty$, we must have $A = 0$, and we get a logistic equation. The solution for N is then

$$N(t) = \frac{1}{\alpha}P'(t) = \frac{1}{2\alpha}\cosh^{-2}\left(\frac{t - t_0}{2}\right). \tag{27}$$

This function is not dissimilar to a Gaussian bell curve with a maximum for $t = t_0$. As expected, small α gives large maximum population.

If one were to adapt the model to actual data, one could start with the inflection points of the function. Let

$$N_0(\tau) = \cosh^{-2}(\tau/2). \tag{28}$$

It is easy to show that $N_0''(\tau) = 0$ for $\tau = \pm 2\,\mathrm{atanh}(1/3) = \pm 1.3170\ldots$ Hence, if we set $\tau_0 = -2\,\mathrm{atanh}(1/3)$, we find

$$\begin{aligned} N_0(\tau_0) &= \frac{2}{3}, \\ N_0'(\tau_0) &= \frac{2}{9}\sqrt{3}. \end{aligned} \tag{29}$$

The population of the Earth is claimed to have passed the inflection point $N''(t_s) = 0$ in 1996, when we, according to my sources, had

$$\begin{aligned} N(t_s) &= 5.75 \times 10^9 \text{ individuals}, \\ \frac{dN}{dt}(t_s) &= 8.3 \times 10^7 \text{ years}^{-1}. \end{aligned} \tag{30}$$

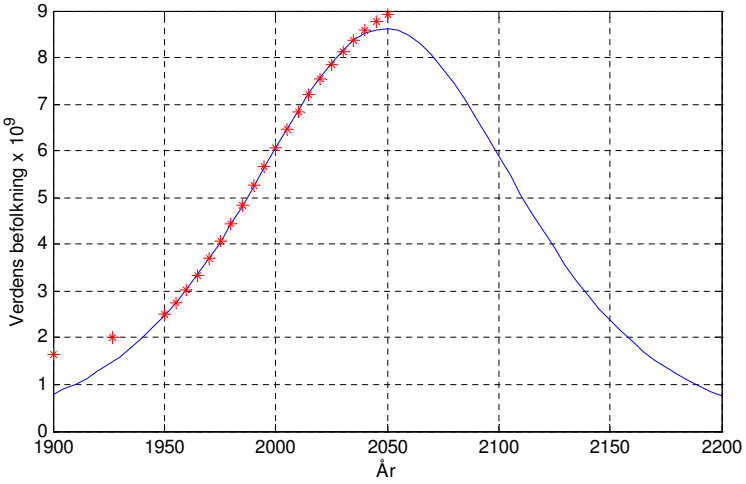

Figure 3: The development of the world's population for a logistic model with poisoning from contaminants (full line). Actual population growth and UN forecast (stars).

By letting

$$N^*(t^*) = N_{\max} N_0 \left(\frac{t^* - t_{\max}}{B} \right), \tag{31}$$

we obtain

$$\begin{aligned} N_{\max} &= \frac{3}{2} 5.75 \times 10^9 \ = 8.62 \times 10^9, \\ B &= \frac{8.62 \times 10^9}{8.3 \times 10^7} \frac{2}{9} \sqrt{3} \text{ years} = 40 \text{ years} \\ t_{\max} &= 1996 + 1.3170 \times 40 = 2049. \end{aligned} \tag{32}$$

In figure 3, $N^*(t^*)$ is plotted against year, and it is compared to data and predictions (created by the UN) which are available on *Wikipedia*. As expected, the curves coincide between 1950 and 2040, but one can clearly not trust such a model too much.

4 COMPETITION FOR THE SAME RESOURCES

In 1934, the Russian biologist Georgyi Frantsevitch Gause (1910–1986) wrote the book *The Struggle for Existence*, where he formulated *The Competitive Exclusion Principle*, which states that two species cannot coexist in the long term if they compete for the same limited resource (applicable to the current situation in the Middle East?). A simple model which reflects such a situation is the following:

$$\frac{1}{N_1^*} \frac{dN_1^*}{dt^*} = r_1 \left(1 - \alpha N_2^* \right), \tag{33}$$

$$\frac{1}{N_2^*} \frac{dN_2^*}{dt^*} = r_2 \left(1 - \beta N_1^* \right). \tag{34}$$

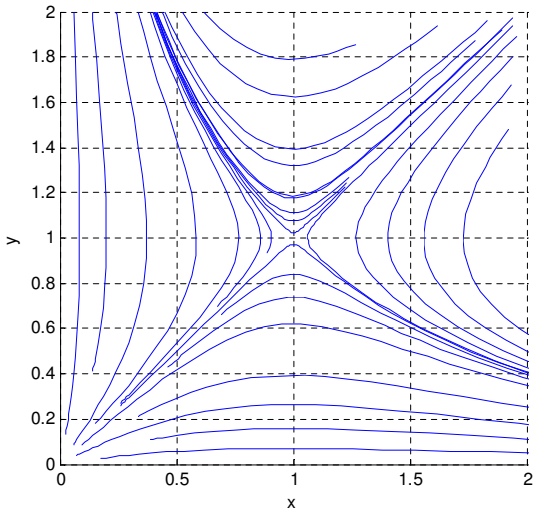

Figure 4: Numerical solutions for Gause's model with $\varepsilon = 1$.

Two obvious time scales appear, namely $1/r_1$ and $1/r_2$. If the difference between the time scales is large, this will be a stiff system, with behaviour characteristic for singular perturbation. The scaling is easy, and leads to a dimensionless system of the form

$$\begin{aligned} \frac{dx}{dt} &= x\,(1-y)\,, \\ \frac{dy}{dt} &= \varepsilon y\,(1-x)\,, \end{aligned}$$

where the parameter ε expresses the relationship between the time scales. From the symmetry of the equations, we see that all conclusions reached about small ε can be rewritten to corresponding statements for large ε. Thus, it is enough to consider $\varepsilon < 0 \leq 1$. The system has equilibrium points in (1,1) which is a saddle point, and (0,0) which is an unstable node. If one of the populations is 0, the other grows without limits, so the model is not especially realistic.

A numerical solution of the system for $\varepsilon = 1$ is shown in figure 4. We see that the first quadrant splits into four areas delimited by the coordinate axes and two curves crossing each other in (1,1). These curves are called *separatrices*. Systems close to the separatrix $y = x$ live dangerously: A small disturbance may cause the movement to jump over to the other side, and we get a fundamentally different development of the population.

It is easy to find an implicit equation for the trajectories by dividing the equations by each other and separating the variables:

$$\left(\frac{1}{y} - 1\right) dy = \varepsilon \left(\frac{1}{x} - 1\right) dx. \tag{35}$$

This gives all non-trivial trajectories, expressed as

$$ye^{-y} = C\left(xe^{-x}\right)^{\varepsilon}\,,\ 0 < \varepsilon \leq 1, \tag{36}$$

where C is a positive constant. For given values of ε and C we see from xe^{-x} that there exist four pairs of solutions. Pairs of these lie on the same trajectories. The maximum value of the

left-hand side is e^{-1}. If $\max_x C(xe^{-x})^\varepsilon > e^{-1}$, i.e. $C > e^{\varepsilon-1}$, we describe the trajectories to the right and left of the equilibrium (1,1). Otherwise, we describe the trajectories over and under (1,1). The separatrices are given implicitly by

$$ye^{-y} = e^{\varepsilon-1}\left(xe^{-x}\right)^\varepsilon. \tag{37}$$

Corresponding models have also been made for warfare. One should take care when making conclusions from such primitive models, as there exist systems in nature that apparently violate Gause's principle (among others, plankton, according to Wikipedia).

5 THE LOTKA–VOLTERRA EQUATIONS

The Lotka–Volterra equations, also known as the predator-prey equations were formulated by Alfred J. Lotka and Vito Volterra independently, around 1925. Since the equations are described in detail in most books about non-linear differential equations, the presentation below is very brief. In the same way as the equations above, they can be written as a system

$$\frac{1}{N_1^*}\frac{dN_1^*}{dt^*} = r_1\left(1-\alpha N_2^*\right), \tag{38}$$

$$\frac{1}{N_2^*}\frac{dN_2^*}{dt^*} = r_2\left(-1+\beta N_1^*\right), \tag{39}$$

where N_1^* is the prey density and N_2^* is the predator density. As in the preceding paragraph, after scaling the system attains the form

$$\frac{dx}{dt} = x\left(1-y\right), \tag{40}$$

$$\frac{dy}{dt} = \varepsilon y\left(-1+x\right), \tag{41}$$

with equilibria in (1,1) and (0,0) independent of the size of ε. The first one is a centre, and the second a saddle point. The trajectories are shown in figure 5.

Here, the implicit equation for the trajectories is

$$ye^{-y}\left(xe^{-x}\right)^\varepsilon = C, \quad 0 < C. \tag{42}$$

Since the left-hand side is limited from above by $e^{-1-\varepsilon}$, we must have $C < e^{-1-\varepsilon}$. For a given possible value of C, x will also be restricted to the interval around $x = 1$ which satisfies $(xe^{-x})^\varepsilon \geq Ce$. We get a similar interval for y defined by $ye^{-y} \geq Ce^\varepsilon$. Thus, the trajectory is inside a rectangle which is defined by equality of the respective inequalities. The shape of the rectangle is determined by ε and C. When C approaches its maximum value, the trajectories shrink towards $(1,1)$ and become approximately elliptical with centre $(1,1)$. When $C \ll 1$ the trajectory shifts towards the coordinate axes, unless $y \approx -\varepsilon x - \ln(C)$, which is the trajectory far away from the coordinate axes. The orbit shape is roughly triangular.

Limitations in growth as in the logistic equation leads to scaled equations of the form

$$\frac{dx}{dt} = x\left(1-y-\alpha x\right), \tag{43}$$

$$\frac{dy}{dt} = \varepsilon y\left(-1+x-\beta y\right) \tag{44}$$

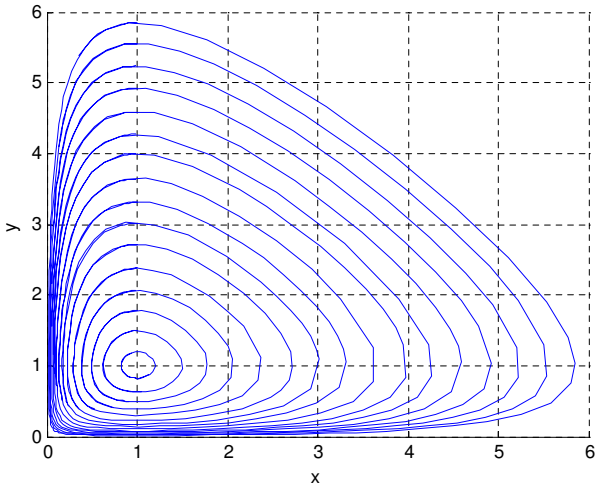

Figure 5: Paths of the Lotka–Volterra system when $\varepsilon = 1$ calculated numerically. The numerical solution goes around the orbits several times and does not connect perfectly.

with equilibria in the origin and (x_0, y_0) where

$$x_0 = \frac{1+\beta}{1+\alpha\beta}\ ,\ y_0 = \frac{1-\alpha}{1+\alpha\beta}. \tag{45}$$

We leave to the reader to show that (x_0, y_0) becomes a stable focus when α and β are small. What else happens?

If we return to the original Lotka–Volterra equations and utilize the fact that the trajectories are periodic with period T, we get

$$\frac{1}{r_1}\int_{t^*=0}^{T} \frac{dN_1^*}{N_1^*} = \int_{t^*=0}^{T} \left(1-\alpha N_2^*\right) dt^* = T - \alpha \int_{t^*=0}^{T} N_2^*\, dt^*, \tag{46}$$

and correspondingly for the other equation. But, since

$$\frac{1}{r_1}\int_{t^*=0}^{T} \frac{dN_1^*}{N_1^*} = \frac{1}{r_1}\left(\ln(N_1^*(T) - \ln(N_1^*(0)\right) = 0, \tag{47}$$

so

$$\frac{1}{T}\int_{t^*=0}^{T} N_2^*\, dt^* = \frac{1}{\alpha}, \tag{48}$$

$$\frac{1}{T}\int_{t^*=0}^{T} N_1^*\, dt^* = \frac{1}{\beta}. \tag{49}$$

The average levels become equal to the values in the equilibrium point.

If one studies the predator-prey models somewhat more in depth, one will see that the Lotka–Volterra equations are special. Typically, such models will have stable equilibrium points. If one

observes oscillations in nature, these will not be periodic solutions about a neutral equilibrium point, but rather so-called *stable limit cycles* which are more stable to perturbations.

Delays in the Lotka–Volterra models have also been studied, and unlike ordinary 2-dimensional ODE systems, such equations can actually have *chaotic* behaviour.

It is also possible to study what hunting means to a Lotka–Volterra system. If we assume constant capture relative to population, this can be modelled as

$$\frac{dN_1^*}{dt^*}\frac{1}{N_1^*} = r_1\left(1-\alpha N_2^*\right) - f_1, \tag{50}$$

$$\frac{dN_2^*}{dt^*}\frac{1}{N_2^*} = r_2\left(-1+\beta N_1^*\right) - f_2. \tag{51}$$

Since we can write

$$\frac{dN_1^*}{dt^*}\frac{1}{N_1^*} = (r_1-f_1)\left(1-\frac{r_1\alpha}{r_1-f_1}N_2^*\right), \tag{52}$$

$$\frac{dN_2^*}{dt^*}\frac{1}{N_2^*} = (r_2+f_2)\left(-1+\frac{r_2\beta}{r_2+f_2}N_1^*\right), \tag{53}$$

we see that as long as the parameters are constant, the behaviour will be like for a Lotka–Volterra system with modified parameters. In particular,

$$\frac{1}{T}\int_{t^*=0}^{T} N_2^*\,dt^* = \frac{1}{\alpha}\frac{r_1-f_1}{r_1}, \tag{54}$$

$$\frac{1}{T}\int_{t^*=0}^{T} N_1^*\,dt^* = \frac{1}{\beta}\frac{r_2+f_2}{r_2}. \tag{55}$$

The model is not necessarily realistic if one looks at the average catch per time unit:

$$F_1 = \left\langle N_1^*\right\rangle f_1 = \frac{1}{\beta}\frac{r_2+f_2}{r_2}f_1, \tag{56}$$

$$F_2 = \left\langle N_2^*\right\rangle f_2 = \frac{1}{\alpha}\frac{r_1-f_1}{r_1}f_2. \tag{57}$$

We observe that if we just catch predators ($f_1 = 0$), we will be able to capture an unlimited amount, without the average level changing. In contrast, the average of the prey population will grow (!).

There is much more that could be studied for such equations, *e.g.* the behaviour for time dependent catch with variations that are long and short relative to the period of the stock variations.

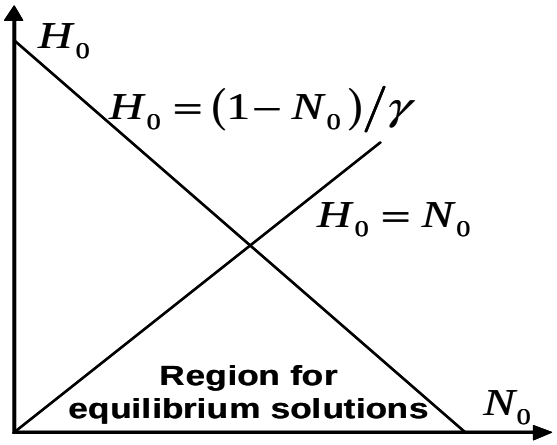

Figure 6: All equilibrium solutions lie within the shaded area when f_N and f_H are between 0 and 1.

6 WHALES AND KRILL

Around 1980, R. M. May made a model for the whale-krill system in Antarctica, where N^* is the krill population and H^* the whale population:

$$\frac{dN^*}{dt^*}\frac{1}{N^*} = r\left(1-\frac{N^*}{K_N}\right) - a_2 H^* - u_N F_N, \tag{58}$$

$$\frac{dH^*}{dt^*}\frac{1}{H^*} = q\left(1-\frac{H^*}{\alpha N^*}\right) - u_H F_H \tag{59}$$

As seen, the maximum sustainable level of the whale stock is proportional to the krill level. The growth rates r and q are clearly quite different. Thus, we expect $1/r \ll 1/q$, and $\varepsilon = q/r \ll 1$. If we scale based on the time scale for changes in the whale population, we end up with the following singularly perturbed system:

$$\varepsilon \dot{N} = N\left(1 - N - \gamma H - f_N\right), \tag{60}$$

$$\dot{H} = H\left(1 - H/N - f_H\right) \tag{61}$$

We assume that f_N and f_H are constants between 0 and 1. Then the model has a stationary point in

$$N_0 = \frac{1 - f_N}{1 + \gamma(1 - f_H)}, \tag{62}$$

$$H_0 = \frac{(1 - f_N)(1 - f_H)}{1 + \gamma(1 - f_H)}, \tag{63}$$

and all singular points lie inside a triangle, as pointed out in figure 6.

The equilibrium solutions are stable focuses, as shown in figure 7. When ε is small, the solutions bear the characteristic sign of *singular perturbation*.

This behaviour is not so striking in areas where $N = O(\varepsilon)$ or $N \ll H$, since one then also will have other small terms in the equations. If the population of whales is lower than the equilibrium

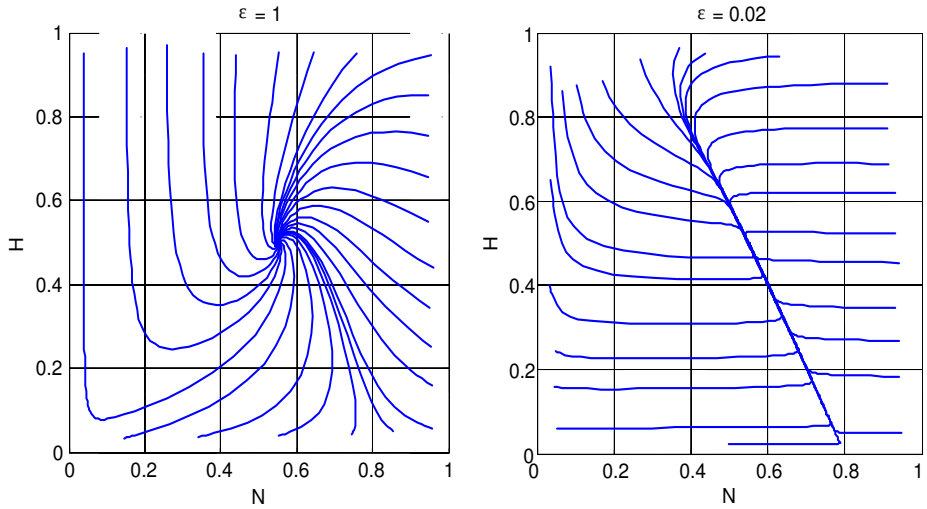

Figure 7: Phase plot of the whale-krill system with $f_N = 0.2$ and $f_H = 0.1$, $\gamma = 0.5$ with $\varepsilon = 1$ to the left, $\varepsilon = 0.02$ to the right.

level, one sees how the system quickly approaches a "quasi-static equilibrium", approximately given by $1 - N - \gamma H - f_N = 0$, and then follows it towards the equilibrium point. Obviously, this is due to the krill population reacting quickly compared to the whale stock. PM7INN

For a given catch rate, the amount caught per time unit is

$$P_N = f_N N_0 = \frac{f_N(1 - f_N)}{1 + \gamma(1 - f_H)}, \tag{64}$$

$$P_H = f_H H_0 = \frac{f_H(1 - f_H)(1 - f_N)}{1 + \gamma(1 - f_H)} \tag{65}$$

We see that the maximum amount of krill we can take out is given by $f_N = 1/2$ whatever the catch of whales, while the maximum amount of whales is $f_H = \left(1 + \gamma\sqrt{1 + \gamma}\right)/\gamma$. It is not possible to increase whaling by fishing krill, while it of course is beneficial to catch whales to be able to catch more krill.

It is easy to imagine economic issues related to such a model. If one catches both whales and krill, one is interested in maximizing profits. If the prices are p_N and p_H, one would like to maximize economic return per time unit, $p_N P_N + p_H P_H$. At the same time there's a limit to the weight of catch over a period T that can be transported with available tonnage, $(w_N P_N + w_H P_H)t \leq L$.

If catch rates are proportional to the inverse of the populations, i.e. a fixed quantity is fished per unit of time irrespective of the size of the stocks, the equations are more cumbersome. Any equilibrium solutions can be found from

$$N\left(1 - N - \gamma H - a_N/N\right) = N - NH^2 - \gamma HN - a_N = 0, \tag{66}$$

$$H\left(1 - H/N - a_H/H\right) = H - H^2/N - a_H = 0. \tag{67}$$

If we assume that $\gamma = 1$ and that we only catch whales, we deduce that

$$N_0 = 1 - H_0,$$

$$H_0 = \frac{1 + a_H \pm \sqrt{(1 + a_H)^2 - 8a_H}}{4}. \tag{68}$$

In other words, if $0 < a_H < 3 - 2\sqrt{2}$, we have two equilibrium points. We leave details to the reader.

An important aspect of such models is whether they are stable against disturbances, and how well the catch can be controlled by decree, etc. As one understands, there is unlimited potential for both mathematical, numerical and economic analysis.

7 SPECIES THAT BENEFIT FROM EACH OTHER

There are many examples of systems where the species have mutual benefits of each other. Plants that depend on pollinators is one such example. Pollinators on the other hand, receive nectar from the plant. In some situations the pollinators are specially adapted to one type of plants (or perhaps the reverse is true?).

It is easy to see that the following simple model, where P is plant population and B is the pollinator, leads to an absurdity:

$$\frac{dB^*}{dt^*}\frac{1}{B^*} = (-1 + \alpha P^*), \tag{69}$$
$$\frac{dP^*}{dt^*}\frac{1}{P^*} = \varepsilon(-1 + \beta B^*). \tag{70}$$

It is a bit more surprising that

$$\frac{dB^*}{dt^*}\frac{1}{B^*} = (-1 + \alpha P^* - \gamma B^*), \tag{71}$$
$$\frac{dP^*}{dt}\frac{1}{P^*} = \varepsilon(-1 + \beta B^* - \delta P^*) \tag{72}$$

does not work either.

The following model, which is taken from May (1978), starts with a logistic equation for the pollinators, where the sustainable population level depends on the plant population:

$$\frac{dB^*}{dt^*} = rB^*\left(1 - \alpha\frac{B^*}{P^*}\right). \tag{73}$$

The plant quantity in turn satisfies an equation of the form

$$\frac{dP^*}{dt^*} = -qP^* + I\frac{\hat{B}}{\hat{B} + C}, \tag{74}$$

where $\hat{B}$ is the effective density of pollinators. If there are very few plants, it is not certain that the pollinators can find plants ($\hat{B} \ll B^*$), while if there are many plants, all pollinators find enough plants ($\hat{B} \approx B^*$). A possible model for $\hat{B}$ can then be

$$\hat{B} = B^*\frac{P^*}{P^* + D}. \tag{75}$$

Together, this gives the equations

$$\begin{aligned}\frac{dB}{dt}\frac{1}{B} &= 1 - a\frac{B}{P},\\ \frac{dP}{dt}\frac{1}{P} &= -\varepsilon + b\frac{B}{BP + P + 1},\end{aligned} \tag{76}$$

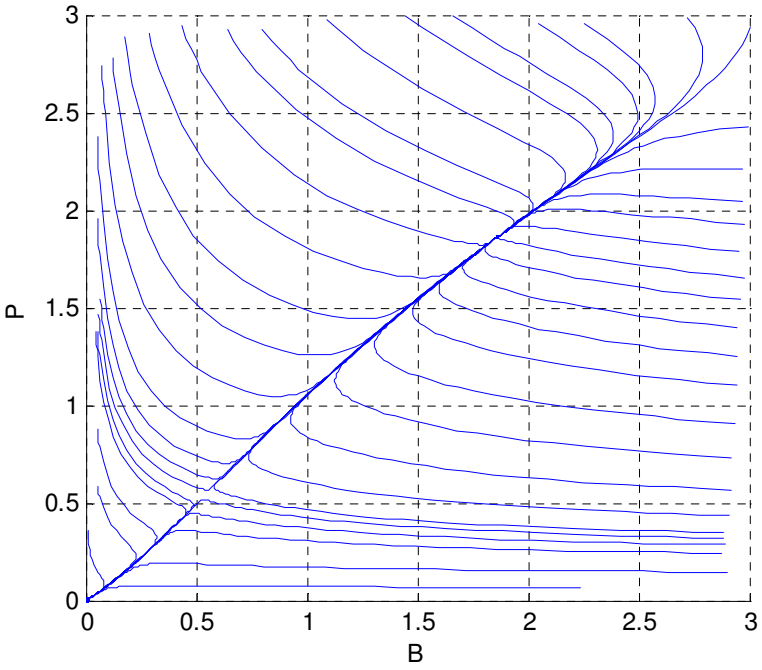

Figure 8: Phase diagram for pollinators (B) and plants (P) for a situation where we have one stable and one unstable equilibrium point.

where we have scaled in the following way:

$$P^* = DP,\ B^* = CB,\ \ a = \alpha C/D,\ \ b = I/(rD),\ \ t^* = tr,\ \ \varepsilon = q/r. \tag{77}$$

By solving for the equilibria, we end up with

$$B_0 = \frac{1}{2a\varepsilon}\left(b - \varepsilon a \pm \sqrt{(b-\varepsilon a)^2 - 4a\varepsilon^2}\right), \tag{78}$$

$$P_0 = aB_0. \tag{79}$$

There will be two equilibria in the first quadrant if $0 < b - \varepsilon a,\ \ 0 < (b-\varepsilon a)^2 - 4a\varepsilon^2$, or

$$0 < (I - q\alpha C)/D, \tag{80}$$

$$0 < \frac{I - q\alpha C)^2 - 4\alpha q^2 CD}{D^2}. \tag{81}$$

Figure 8 shows a set of parameters giving two equilibria points ($\varepsilon = 1/2, a = 1, b = 1.7$).

If the system is left of the separatrix, it dies out. Isolated systems of this kind are common only in areas with stable climatic conditions. According to the reference, there are e.g. no such ant-plant systems north of 24°, no nectar-eating bats (!) north of 33°, or orchid-bees north of 24° in America.

8 AFTERWORD

The mathematical theory underlying the models we have seen is a part of what is called *dynamical systems*. This is a field with large and interesting activity that not only includes differential

equations, but also time-discrete and stochastic models. In special cases, like for some systems of ordinary differential equations with more than two variables, discrete models, or equations with delay, the solutions exhibit so-called *chaotic* behaviour. In nature, there are systems that seem to be fairly robust to perturbations (*e.g.* exhibit stable limit cycles), and the completely opposite behaviour (*e.g.* equations for atmospheric turbulence, which controls the weather).

9 EXERCISES

9.1 Equilibrium, Stability, and Bifurcations

Problem 1

Find the equilibrium solutions of the following differential equations and determine whether they are stable or unstable:

$$\begin{array}{ll} (i) & \frac{du}{dt} = u^2(u^2-1), \\ (ii) & \frac{du}{dt} = \left(u-u^2\right)(u-\mu)\,,\ u \geq 0, \mu \geq 0. \end{array} \tag{82}$$

Problem 2

Explain that, in order to determine the stability of an equilibrium point u_0 for the model $du/dt = f(u)$, it is usually sufficient to consider the sign of $f(u)$ around u_0. Apply this when making a sketch of the bifurcation diagrams indicating stable and unstable equilibrium points in the (μ, u)-plane for the models

$$\begin{array}{ll} (i) & \frac{du}{dt} = \left(\mu + u^2 - 2u - 1\right)(\mu + u)\,, \\ (ii) & \frac{du}{dt} = u(9-\mu u)(\mu + 2u - u^2). \end{array} \tag{83}$$

(It is not necessary to derive the exact expressions for the equilibrium points or consider points requiring higher order analysis).

Problem 3

A certain population develops according to the equation of logistic growth. For small populations, there is some possibility that individuals may die before they have met a partner, and this can be modelled by modifying the logistic model as follows,

$$\frac{dy}{dt} = y - y^2 - \mu y,\ \mu \geq 0. \tag{84}$$

(a) Determine the equilibrium solutions of Eq. 84 and determine whether they are stable or unstable.

For larger populations, this modification will be less important. The following model has therefore been proposed:

$$\frac{dy}{dt} = y\left(1 - \frac{y}{M} - \frac{\mu}{1+y}\right), \tag{85}$$

Here we assume that $M > 1$ and, as above, that $\mu > 0$.

(b) Sketch the bifurcation diagram for the modified model and determine stable and unstable equilibrium solutions. Investigate what happens when the population is stable and positive, and μ grows slowly past the value $\frac{(M+1)^2}{4M}$.

Problem 4

The atmosphere of the Earth, the upper layer of the ocean (down to about 20m), and the upper few meters of the crust make up a mass M of about 3×10^{18}kg. It is this mass that continuously

receives heat from the sun and looses heat by radiation into space. The simplest model for Earth's mean temperature may therefore be written

$$\frac{Mc}{4\pi R^2}\frac{dT}{dt} = Q_0 - \sigma T^4, \tag{86}$$

where R the radius of the Earth, c is the mean specific heat, Q_0 is the mean absorbed heat radiation from the sun per area unit, and σ is Stefan-Boltzmann's constant. We write the equation as

$$C\frac{dT}{dt} = Q_0 - \sigma T^4, \tag{87}$$

and apply $Q_0/\sigma = (287\text{K})^4$ and $C/\sigma = 6 \times 10^9 \text{days}\times\text{K}^3$ in the following.

(a) Find the stationary temperature, show that it is stable, and the time (in days) it takes for some perturbation of the equilibrium temperature to die out.

The amount of absorbed radiation from the sun depends on, among others things, the color of Earth's surface (also called *albedo*). During ice ages, when a relatively larger part of Earth is covered with ice and snow, the mean absorbed energy will be less than during periods with higher temperatures. It has therefore been suggested to modify the term Q_0 to

$$Q_0 + Q_a \tanh\left(\frac{T - T_0}{T_n}\right), \tag{88}$$

where $Q_0 - Q_a$ represents ice ages, $Q_0 + Q_a$ warm periods, and T_0 is the equilibrium temperature from (a). The temperature T_n controls the width of the transition range.

(b) Discuss the modified model, and explain what can happen if

$$T_n < \frac{Q_a}{4Q_0}T_0 \tag{89}$$

and Q_0 varies.

Hint: Make a sketch and recall that

$$\begin{aligned}
&\tanh(x) \underset{x\to-\infty}{\longrightarrow} -1,\\
&\tanh(x) = x + O\left(x^3\right), \text{ for small } x\text{-s},\\
&\tanh(x) \underset{x\to\infty}{\longrightarrow} 1.
\end{aligned}$$

9.2 Population Models

Problem 5

(a) Let $0 < \alpha < r$, $N_0 > 0$, and consider the following equation:

$$\frac{dN^*(t^*)}{dt^*} = rN^*(t^*)\left(1 - \frac{N^*(t^*)}{K}\right) - \alpha N^*(t^*), \quad t^* > 0, \tag{90}$$

$$N^*(0) = N_0. \tag{91}$$

What could the problem in Eq. 91 model? Find reasonable scales for this problem when $N_0 \gg K$ and $\alpha < r$.

(b) Let $\varepsilon, \alpha, \beta > 0$, $\alpha, \beta < 1$, and consider the following scaled system of equations for two interacting populations x and y:

$$\frac{dx}{dt} = x(1-x) - \alpha xy, \tag{92}$$

$$\varepsilon \frac{dy}{dt} = y(1-y) - \beta xy. \tag{93}$$

Discuss the properties of the model in terms of the parameters ε, α, and β. Find the equilibrium points and determine their stability when $\beta = 0$.

Problem 6

The following scaled model has been proposed for the population of females (F) and males (M) in a fish population:

$$\dot{F} = -F + \gamma \left(1 - e^{-M}\right) F, \tag{94}$$

$$\dot{M} = -M + \gamma \left(1 - e^{-M}\right) F \tag{95}$$

(γ is a positive constant).

(a) Explain some of the properties of the model. For example,

- What happens if F or M is 0?
- Is there a difference in the birth rate for females and males?
- How are the equations when M is small and when M is large, and how are the conditions reflecting this?

(b) What kind of point is the stationary point $(0, 0)$?

(c) Show that, if $\gamma > 1$, there is another stationary point (F_0, M_0) where $F_0 = M_0$.

(d) Find the type of the point in **(c)**.

(e) Is the model really reasonable?

Problem 7

The Eurasian elk or moose (*Alces alces*) is the largest species in the deer family. Each year over 8000 moose calves and adults are caught during the hunting season in the forests around Trondheim. Consider the following simple model for the moose population, P:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(\frac{P}{m} - 1\right), \quad 0 < m < M. \tag{96}$$

(a) What properties does the model try to explain, and what are the equilibrium populations? Determine, using linear stability analysis, whether they are stable or unstable. Make a simple sketch showing the solutions of Eq. 96.

A simplified model, also including the hunter population (J) has, after scaling, the form

$$\frac{dP}{dt} = P(1-P) - J,$$

$$\frac{dJ}{dt} = -\frac{J}{2} + JP. \tag{97}$$

(b) Determine the equilibrium points for (97) and of what kind they are (or appear to be).

(c) Show that the solution of

$$h\left(P,J\right)=J-3P(1-P)/2=0 \tag{98}$$

defines a possible path for the model in (97).

(*Hint*: Show that ∇h along this path is always orthogonal to the direction of the motion defined by $(dP/dt, dJ/dt)$).

(d) The point $(P_c, J_c) = \left(\frac{1}{2}, \frac{1}{4}\right)$ for (97) could be a center. Shift the origin by introducing

$$x = P - 1/2, \tag{99}$$

$$y = J - 1/4, \tag{100}$$

and show that the system has paths which are symmetric about the new y-axis, and therefore that the point *is* really a center.

(e) Summarize the qualitative properties of the model in (97).

Problem 8

In a population consisting of M_0 individuals, an influenza infection is transferred by sick persons ($S_0^*(t)$ individuals) meeting susceptible persons ($M_0 - S_0^*(t)$ individuals). The number of people infected per time unit is proportional to the probability that sick persons meet susceptible persons. The proportionality constant, r is called the *infection rate* (per individual). After a while, the sick individuals recover.

(a) The following dynamic model has been proposed for the number of sick persons in the population if one assumes that none are immune:

$$\frac{dS^*}{dt^*} = rS^*\left(M_0 - S^*\right) - \alpha S^*. \tag{101}$$

Consider a simplified equation in the early stages of the epidemic, and find suitable scales for the variables. The infection rate can be controlled by vaccines and other restrictions by the health authorities. Discuss the stationary solutions of the scaled equation in the light of the size of r.

A more realistic model takes into account that those who have been cured will be immune, at least for some time after the illness. The following (scaled) mathematical model has been proposed for the number of sick, $S(t)$, and immune, $I(t)$, persons:

$$\frac{dS}{dt} = S\left(1 - I - S\right) - \lambda S, \tag{102}$$

$$\frac{dI}{dt} = \lambda S - \mu I. \tag{103}$$

(b) State the region in the (S,T)-plane for physically acceptable solutions, and the meaning of the parameters μ and λ. This dynamical system has an obvious stationary point. When is this point stable? It is not necessary to study limit cases.

(c) Determine for which other values of λ and μ the system has another, physically acceptable stationary point, (S_0, I_0). Show that this point moves on a line segment if λ is kept fixed while μ varies.

(d) Linearize the system in Eq. 103 around (S_0, I_0) by introducing $S(t) = S_0 + x(t)$, $I(t) = I_0 + y(t)$ and show that the matrix $\mathbf{A}$ in the linearized system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{x} = [x, y]'$, has the form

$$\mathbf{A} = \begin{bmatrix} -S_0 & -S_0 \\ \lambda & -\mu \end{bmatrix}. \tag{104}$$

Determine whether (S_0, I_0) is stable or unstable.

Problem 9

The cell density, n^*, in a part of the body may be modelled as

$$\frac{dn^*}{dt^*} = \alpha n^* - \omega n^*, \tag{105}$$

where α is the birth rate and ω the death rate of the cells. In order to prevent that production of cells runs astray, the cells produce a chemical called *inhibitor,* which dampens uncontrolled growth. The inhibitor has density c^* and works by changing the birth rate to

$$\alpha = \frac{\alpha_0}{1 + c^*/A}. \tag{106}$$

The production of the inhibitor is proportional to n^*, whereas the destruction of the inhibitor is proportional to its density,

$$\frac{dc^*}{dt^*} = \beta n^* - \delta c^*. \tag{107}$$

The system consisting of the equations 105 and 107 has a time scale ω^{-1} connected to the breakdown of the cells, and a time scale δ^{-1} connected to the breakdown of the inhibitor. It is known that $\omega^{-1} \gg \delta^{-1}$.

(a) Scale the system by applying ω^{-1} as the time scale and A as a scale for c^*. Show that the system with a certain scale for n^* may then be written

$$\begin{aligned} \dot{n} &= \left(\frac{\kappa}{1+c} - 1\right) n, \\ \varepsilon \dot{c} &= n - c. \end{aligned} \tag{108}$$

What is the meaning of ε and κ? What can be said about the size of ε, and what is such a system called? Determine what kind of equilibrium point the trivial equilibrium point $(0, 0)$ is.

Here and below we assume that κ is larger than 1.

(b) Determine the path and the equation for the motion of the outer solution of Eq. (108) to leading order ($O(1)$). Show, without necessarily solving the differential equation, that all motion on this path converges to an equilibrium point for the full system.

(c) Determine, to leading order, the inner solution of (108) by introducing a new time scale. Then determine a uniform, approximate solution (It is not possible to solve the equation in (b) explicitly).

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11 APPENDIX: EQUATION SOLVER IN MATLAB

There are many interactive solvers for two-dimensional ODE systems on the *Internet*. Below is a simple solver in MATLAB for testing out the systems discussed in the text. The solver consists of two m-files. The first one is the program itself, while the second computes the derivatives (right-hand side) of the equations. See the documentation of MATLAB for more information about `ode45`. The MATLAB function `ginput` makes it possible to point in the plot with the mouse to set the initial values (Text after `%` in the code are comments).

```
tspan = [0 10];           % Define a time interval
axis([-3 3 -2 4]); grid;  % Define the axes
hold on                   % Lock the coordinate system
for loop = 1:100          % Allow for up to 100 repetitions
  x0 = ginput(1);         % Pick the initial value by pointing with the mouse
  [t,x]=ode45('xder',tspan,x0);  % Integrate the equation
  plot(x(:,1),x(:,2))     % Draw the path
end;

function xd=xder(t,x)
s = x(1);
c = x(2);
xd(1)=-s*(1-c)+c;
xd(2)=s*(1-c)-c+(1-c).^2;
xd = xd';  % column vector
```

Figure 9 shows an example of how the trajectories look for the equation above with a random selection of initial values.

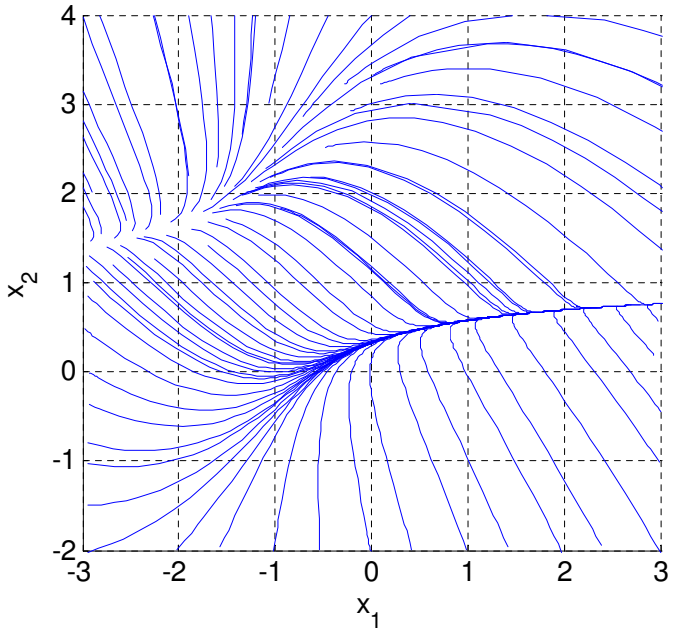

Figure 9: Paths for the equation defined in the Matlab file `xder.m` . Starting points are selected by the mouse-pointer, making it possible to examine some areas of the chart in more detail.

TMA 4195
Mathematical Modeling

Supplementary Notes, Part 3:

Modeling Based on Conservation Principles

Autumn 2011

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Contents

1 INTRODUCTION

In this part we shall consider concepts and approaches for models based on *conservation principles*. Some of the material will be relatively easy, at least for students with some background in fluid and continuum mechanics. Nevertheless, it is important to see that the principles are generally valid and applicable for modeling in many other contexts than traditional mechanics.

It is difficult to discuss conservation principles without getting involved with partial differential equations, but we shall mostly be interested in the qualitative theory and the general behavior of the solutions, and not very specific analytical or numerical solution methods. Below, the term *fluid* is used both for liquids and gases. A *first aid* course about first-order quasi-linear partial differential equations is found in an appendix. It is designed for readers without any knowledge of partial differential equations beyond calculus.

In many models we apply continuous distributions or densities in space where the phenomenon we consider is discrete. Typical examples could be the density of bacteria, people, cars, and the like. In such cases, the models implicitly assume a kind of *continuum hypothesis* similar to the one in continuum mechanics. However, such an approach has obvious limitations, and it is important to be aware what the models are really good for. It leads, *e.g.* to absurdities to insist on mathematical definitions based on limits to ∞ or 0. In the physical world "0" is a few orders of magnitudes below and "∞" some orders of magnitudes above where we are located. When we say that "$\Delta x \to 0$", we actually mean that Δx is small compared to the scale where we are located, and not that Δx really goes to 0 in the mathematical sense. This is similar to talking about stationary conditions in time from $-\infty$ and ∞. In elementary particle physics the eternity could well be 10^{-10}s! Scaling considerations that we have covered earlier, help us to assess the reliability of our assumptions.

We should distinguish between establishing a model and solving the equations after they have been formulated in a mathematical model. The latter is the theme of the courses in analytical and numerical solution of ordinary and partial differential equations. Although enthusiasts advertise numerical software that can solve any differential equation, it is still far from that we can leave to the computer to *understand* what is really happening. As numerical tools are becoming more advanced, it is, on the contrary, and increasing demands for mathematical expertise and analysis of the equations. Today, serious customers require that calculations based on numerical models should be documented to be reliable. Only thorough mathematical and numerical analysis, and not least physical and engineering insight, can help with this. Many of the analytical solutions for idealized problems that are known from the theory of partial differential equations are useful in this respect. Scaling arguments show that the so-called fundamental solution of the heat conduction equation has far greater applicability than is usually mentioned in the mathematics courses. In a way, fundamental solutions and other solutions from idealized mathematical situations are the cornerstones that give us insight and set limits.

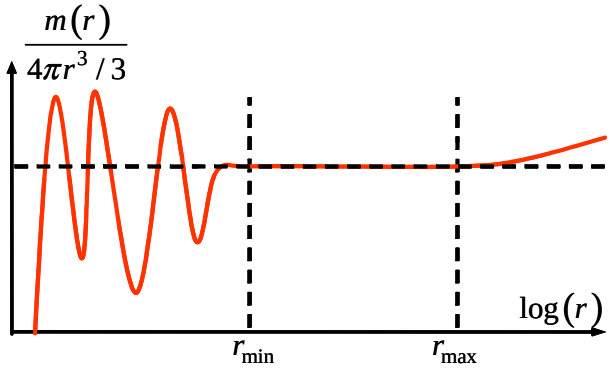

Figure 1: Pragmatic definition of *density* at a point in space: The mass over volume ratio is reasonably stable for $r_{\min} < r < r_{\max}$, but the limit when $r \to 0$ does not exist and makes no sense.

2 BASIC CONCEPTS

2.1 Density

Although we perceive water and air as quite homogeneous and uniform physical materials, we all know that this is only when considered from our own length-scale. If we made an imaginary sphere with radius r and center at $\mathbf{x}$ in air and could calculate the mass within the sphere, $m(r)$, the mathematical definition of the density of the air in the point $\mathbf{x}$ would be

$$\rho(\mathbf{x}) = \lim_{r \to 0} \frac{m(r)}{4\pi r^3/3} \tag{1}$$

If we really were able to perform this experiment, and plot the ratio in a graph as a function of r, we would however see something like in Fig. 1. When r becomes less than $r_{\min}$ (about 10^{-7}m for the air around us), the ratio begins to fluctuate, and it is certainly no sense in talking about a limit when r goes to 0, as we do in mathematics. Conversely, if r is too large, the ratio will no longer be constant because the air inside the sphere is no longer uniform. As you understand, we must add to the definition of density an assumption that we stop the limit process in the right place, and that our definition of the density of air only has meaning for phenomena with a length scale between $r_{\min}$ and $r_{\max}$. As applied mathematicians, we have to bear with density not being particularly "well-defined". This does not create major problems for air and water in most of our daily situations, but for high vacuum technology, $r_{\min}$ may well be of the same scale as the apparatus.

Let us consider some quantity that we describe by a density $\varphi(\mathbf{x}, t)$. The amount within a given closed region R of space may be expressed by the volume integral,

$$M(t) = \int_R \varphi(\mathbf{x}, t)\, dV. \tag{2}$$

Although one immediately think of density as amount per volume unit, there is nothing wrong in defining the density as the amount per of area unit, like 80g/m^2 for ordinary

writing paper. Similarly, for a thin iron wire the useful density becomes weight per unit length.

In the introduction to [11], the authors have two examples which illustrate how it is possible to think of continuous densities in two quite extreme cases. One example discusses how the spiral structure of galaxies can be modeled as density waves in a gravitational plasma in which the galaxies are modeled as a continuum characterized by the mass density of the stars (C. C. Lin was in fact one of the main contributors to this theory). The second example discusses instabilities in the density of amoebas during food shortages, the density is the number of amoebas per area unit, also regarded as a continuous function of the position on the surface they live (this model and analysis of it was developed by the second author, L. A. Segel).

In mathematical modeling we therefore talk about densities in many other situations than we know from mechanics. The density of various foreign substances, such as contaminants in water, is also a relevant example. Within the air pollution modeling, the most advanced mathematical models consider hundreds of different components, each of which is characterized by its density. In addition, the components interact, decay chemically, are transported with the wind and become mixed in the air masses, or simply fall down. Oil reservoir engineering applies complicated mathematical models for tracing various oil and gas components in porous rocks.

The heat or energy content in materials may be expressed as energy per volume unit. This will in the simplest case with constant specific heat be proportional to the temperature. Entropy density appears in models that deal with heat conduction and heat transfer. Some densities lead us into mathematical problems (so-called *singular densities*) which we shall return to below in the section about sources and sinks.

In continuum mechanics we call quantities that passively follows the flow for material variables. The most common material variables in mechanics are mass, momentum, vorticity and energy, which, in a continuous medium, are described by

$$\begin{array}{ll} \rho & \text{mass density} \\ \rho\mathbf{v} & \text{momentum density} \\ \nabla \times \rho\mathbf{v} & \text{Vorticity density} \\ e\rho & \text{energy density} \end{array} \tag{3}$$

where $\mathbf{v}$ is the velocity of the medium. Impurities or other additives that passively follow the flow, are also material variable. The concentration of plankton in the water is therefore a material variable as long as it does not move on their own. On the other hand, a school of herring will usually not count as a material variable!

Within biology it is common, as in the example by L. A. Segel, to operate with continuous density functions of animals, bacteria and plants. This allows one to create models which describe the motion of animal herds, bacterial cultures, the spread of epidemics, and the like. We shall later look at a situation where we model the density of cars along a road as a continuum.

2.2 Flux

Flux is about transport or flow of something. The term has actually various meanings in science, but here it is only connected to the motion. If we stand by a road and watch the cars passing, the average number of cars passing per minute will be what we define the *flux of cars.* Flux includes the direction of the flow, so the flux of cars should be separated into flux to the right, and flux to the left.

We will meet flux in a lot of different situations. To fix ideas, let us consider the flow of some material in space. Standing at a fixed point $\mathbf{x}$ we observe that it passes, but in order to quantify how much is flowing, we put (an imaginary) open small window frame $d\sigma$ into the stream at $\mathbf{x}$ and observe how much is passing through the frame per second. It is convenient to present the measurements as amount per second and area unit, since a window twice as large and with the same orientation will allow twice as much material to pass through. The orientation of the frame is uniquely defined by a fixed normal vector $\mathbf{n}$ attached to the frame. If we change the orientation and hence the normal vector to the opposite direction, the flux changes sign. The maximum amount will flow through the frame if we align $\mathbf{n}$ with the direction of the flow, and this direction is what we define to be the *direction of the flux.* Flux can therefore be most easily described as a *vector field*, $\mathbf{j}(\mathbf{x}, t)$, where the direction of $\mathbf{j}$ indicates the transport direction, and the size, $|\mathbf{j}|$, the amount per time and area unit. Nothing passes through the frame if $\mathbf{n}$ is orthogonal to $\mathbf{j}$. In general, the amount dM that passes through $d\sigma$ (with orientation $\mathbf{n}$) during a time period dt is thus

$$dM = \mathbf{j} \cdot \mathbf{n} d\sigma dt. \tag{4}$$

The total amount flowing out through a surface Σ in space with normal vector $\mathbf{n}$ per unit of time is now given by the surface integral,

$$\frac{dM}{dt} = \int_{\Sigma} \mathbf{j} \cdot \mathbf{n} d\sigma. \tag{5}$$

By means of the *Divergence Theorem* from vector analysis, it is possible to rewrite the surface integrals over the closed surface ∂R of a volume R as

$$\int_{\partial R} \mathbf{j} \cdot \mathbf{n} d\sigma = \int_{R} \nabla \cdot \mathbf{j} dV. \tag{6}$$

We shall later see that some care must be exercised when applying the Divergence Theorem.

For a material variable with density φ, passively following a continuous flow with a velocity vector field $\mathbf{v}$, the flux has a particularly simple form, namely, $\mathbf{j} = \varphi\mathbf{v}$. This follows from Fig. 2, where an amount φdV passes through $d\sigma$ during the time dt. Simple vector calculus gives $dV = d\sigma \cdot |\mathbf{v}| dt \cdot \cos\alpha = \mathbf{v} \cdot \mathbf{n} d\sigma dt$. Therefore, the amount passing through $d\sigma$ per time unit will be

$$Q = \frac{\varphi dV}{dt} = \varphi \mathbf{v} \cdot \mathbf{n} d\sigma = \mathbf{j} \cdot \mathbf{n} d\sigma. \tag{7}$$

The flux for this particular case may be written $\mathbf{j} = \varphi\mathbf{v}$, and the total flow of the material through a surface Σ per time unit is

$$Q = \int_{\Sigma} (\varphi \mathbf{v}) \cdot \mathbf{n} d\sigma. \tag{8}$$

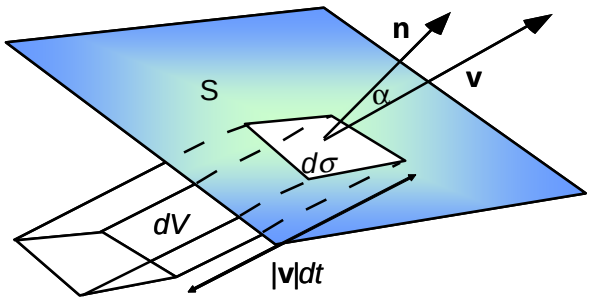

Figure 2: Derivation of the expression for the flux of a material variable.

There exist a lot of different expressions for flux. In a practical modeling situation it may sometimes be difficult to come up with a good model. The expression for the turbulent dispersion and transport of material considered later in this note is not yet fully resolved, despite more than 100 years of active research. In electricity, we have in the general form of Ohm's law that the flux of electric current, $\mathbf{j}$, is given by $\mathbf{j} = \sigma\mathbf{E}$ where σ is the material conductivity and $\mathbf{E}$ electric field strength. In electromagnetism, there electromagnetic radiation carries energy, the energy flux is given by Poynting vector, $\mathbf{P} = \mathbf{E} \times \mathbf{H}$.

We mentioned in the previous section that plankton could be a material variable. Now it is known that plankton to some extent is attracted by light, therefore, will set up a flux directed towards the light.

Of the more curious models of flux, we have the assumption that schools of fish tend to move along the gradient of its *well-being* function, g, such that

$$\mathbf{j} \propto \nabla g. \tag{9}$$

We shall later return to other models of flux, e.g. diffusion generated flux.

2.3 Sources and Sinks

A *source* produces a certain amount of substance per time unit, and we may define a *sink* as a source with negative output, hence we only discuss sources. Mathematically, a study of a source in the point $\mathbf{x}_0$ could be carried out by considering a ball R of radius r around $\mathbf{x}_0$, then computing

$$\int_{\partial R} \mathbf{j} \cdot \mathbf{n} d\sigma, \tag{10}$$

and see what happens when $r \to 0$. Again, we must have the same reservations in mind that we had for definition of density.

One usually distinguishes between *distributed* and *singular* sources. For a distributed source, the limit value of

$$\lim_{r \to 0} \frac{\int_{\partial R} \mathbf{j} \cdot \mathbf{n} d\sigma}{4\pi r^3/3} \tag{11}$$

exists, and when $\mathbf{x}_0$ varies, it defines a function, a so-called *production density* $q(\mathbf{x},t)$ that expresses the production per time and volume unit. The production density is related to divergence of flux field if the flux and its divergence are nice and continuous functions. Applying the Divergence and Mean Value Theorems,

$$\int_{\partial R} \mathbf{j}\cdot\mathbf{n}d\sigma = \int_R \nabla\cdot\mathbf{j}dV = (\nabla\cdot\mathbf{j})(\mathbf{x}')\times 4\pi r^3/3,\ |\mathbf{x}'-\mathbf{x_0}|<r, \tag{12}$$

and thus,

$$\lim_{r\to 0}\frac{\int_{\partial R}\mathbf{j}\cdot\mathbf{n}d\sigma}{4\pi r^3/3} = \lim_{r\to 0}(\nabla\cdot\mathbf{j})(\mathbf{x}') = (\nabla\cdot\mathbf{j})(\mathbf{x_0}). \tag{13}$$

In conclusion,

$$q(\mathbf{x},t) = \nabla\cdot\mathbf{j}(\mathbf{x}) \tag{14}$$

Obviously, for a certain production density, the total production in the volume R during the time from t_1 to t_2, is

$$Q(t) = \int_{t_t}^{t_2}\int_R q(\mathbf{x},t)\,dV\,dt. \tag{15}$$

The simplest singular sources are the *point sources*. For point sources we have, as the name indicates, only production from a single point $\mathbf{x}_0$. This is an idealized situation where, in practice, $q(\mathbf{x},t)$ only differs from zero near $\mathbf{x}_0$. For a ball R around $\mathbf{x}_0$ as considered above,

$$\int_{\partial R}\mathbf{j}\cdot\mathbf{n}d\sigma \tag{16}$$

will stay constant move towards $r\to 0$, or more general be a function of time. If we were to describe a point source at $\mathbf{x}_0$ by a production density, this would be 0 for $\mathbf{x}\neq\mathbf{x_0}$, while the integral over an arbitrary small region containing $\mathbf{x}_0$ would be different from 0. Regular functions with this property do not exist, but as you probably know, one has introduced *generalized functions* (also called *distributions*) to be used in such situations. A point source at $\mathbf{x_0}$ can thus be described using a so-called δ-function,

$$q(\mathbf{x},t) = Q(t)\,\delta_{\mathbf{x}_0}(\mathbf{x}). \tag{17}$$

Here $Q(t)$ denotes the production rate (amount per time unit), and the δ-function states that production occurs in point $\mathbf{x_0}$. In general, the δ-function at $\mathbf{x_0}$ is defined by the property that

$$\int f(x)\,\delta\mathbf{x}_0(\mathbf{x})\,dV = f(\mathbf{x}_0). \tag{18}$$

for all continuous functions.

Point sources are not completely covering the category of singular sources. Often, one needs to model sources located along a surface or a curve in the area to be considered. For example, one can think of a varying heat production along a curve (think of a electrical resistance wire!) or a surface, as indicated in Fig. 3. This also leads into generalized functions, but one can always model such sources as limits of point sources where these

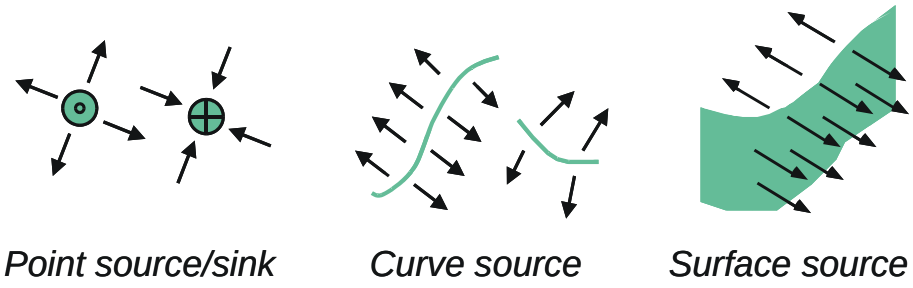

Figure 3: Illustration of various singular sources.

spreads out along a curve or on a surface. We leave to the reader to find out how the above integral, $\int_{\partial R} \mathbf{j} \cdot \mathbf{n} d\sigma$, will behave when $r \to 0$ for curve and surface sources.

In practice, it is often appropriate also to include the singular sources into the production density, which then, in the mathematical sense, becomes a generalized function.

From physics (or distribution theory in mathematics), *dipole* and *quadrupole* sources are also known. We will not meet these in any of the situations we shall consider here.

2.4 The Universal Conservation Law

Let us consider a geometrically closed, imaginary region R with boundary ∂R in space. In modeling, such a region is often called a *control volume*. No part of the boundary needs to be physical, so that material may flow freely through.

In space we have a material with density $\rho(\mathbf{x},t)$ moving with a flux $\mathbf{j}(\mathbf{x},t,\rho)$, dependent of $\mathbf{x}$, t and ρ. For the sources and sinks inside R, we prescribe a generalized production density $q(\mathbf{x},t)$, which in general may contain singular sources. From what we have been through, we see that the rate of change rate in the total amount in R may be written

$$\frac{d}{dt}\int_R \rho(\mathbf{x},\mathbf{t})\, dV. \tag{19}$$

Now, this must be equal to *minus* what is disappearing over the boundary of R per time unit (because of the definition of an *outer* unit normal, $\mathbf{n}$),

$$-\int_{\partial R} \mathbf{j}(\mathbf{x},t,\rho)\cdot\mathbf{n} d\sigma \tag{20}$$

plus what is produced (or disappears) in R per time unit,

$$\int_R q(\mathbf{x},t)\, dV. \tag{21}$$

Altogether, we obtain the equation

$$\frac{d}{dt}\int_R \rho(\mathbf{x},t)\, dV + \int_{\partial R} \mathbf{j}(\mathbf{x},t,\rho)\cdot\mathbf{n} d\sigma = \int_R q(\mathbf{x},t)\, dV. \tag{22}$$

This is called a *conservation law* in *integral form*. The integrals and the derivative with respect to t in the first term exist under fairly general conditions. Otherwise, the conservation law is the mathematical formulation that "nothing can disappear or arise from nothing", – a law of nature with an overwhelming empirical basis!

If ρ and $\mathbf{v}$ are sufficiently smooth functions of t and $\mathbf{x}$, we can move the derivative with respect to t under the integral sign, and otherwise use the Divergence Theorem,

$$\begin{aligned} \frac{d}{dt}\int_R \rho dV &= \int_R \frac{\partial \rho}{\partial t} dV, \\ \int_{\partial R} \mathbf{j}\cdot\mathbf{n} d\sigma &= \int_R \nabla\cdot\mathbf{j} dV. \end{aligned} \tag{23}$$

This means that

$$\int_R \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{j} - \mathbf{q}\right) dV = 0. \tag{24}$$

Such a relation would actually hold for all nice R if ρ and $\mathbf{v}$ are smooth and nice functions in the domain we are considering (Mathematically, it is sufficient that it holds for all spheres in the domain). If $\rho_t + \nabla\cdot\mathbf{j} - q$ is continuous, a result from analysis that says that if Eq. 24 hold,

$$\rho_t + \nabla\cdot\mathbf{j} - q = 0. \tag{25}$$

(Suppose that for a fixed t, $f(\mathbf{x}) = \rho_t + \nabla\cdot\mathbf{j} - q$ is different from 0 at the point $\mathbf{x}_0$. Then f is different from 0 for all $\mathbf{x}$ a neighborhood $N \subset R$ around $\mathbf{x}_0$ since f is continuous. Consequently, the integral of f over this neighborhood is also different from 0, contradictory to the assumption). Eq. 25 thus apply when ρ, $\mathbf{j}$, $\mathbf{v}$ and q are smooth, and this is the conservation law stated in *differential form*.

Since we later will see examples where one cannot move the derivation inside the integration sign, the integral formulation more general and fundamental than the differential formulation.

2.5 Conservation Laws in one Space Dimension

We shall discuss some properties of simple conservation laws, and limit ourselves to a simple one-dimensional situation. The *density*, $\rho(x,t)$, has now the dimension amount per unit length. The *flux* $j(x,t,\rho)$ expresses the amount of material that passes the point x (in the positive x-direction) per time unit, and will in general also be a function of t and ρ. Note that the flux is a *vector* directed along the x-axis. We shall in this discussion for simplicity ignore sources and sinks.

For a finite segment $[A,B]$ of the x-axis, we may, since nothing disappears or is produced in $[A,B]$ write

$$\frac{d}{dt}\int_A^B \rho(x,t)\,dx + j(B,t,\rho(B,t)) - j(A,t,\rho(A,t)) = 0. \tag{26}$$

Note that the boundary unit vector $\mathbf{n}$ is $-\mathbf{i}$ in A and $\mathbf{i}$ in B. If $\rho(x,t)$ is a sufficiently smooth function, we have for all possible subintervals $[a,b]$ of $[A,B]$ that

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \int_a^b \frac{\partial \rho(x,t)}{\partial t}dx = \frac{\partial \rho(\xi,t)}{\partial t}(b-a),\ \xi \in (a,b). \tag{27}$$

Thus, we may write

$$\frac{\partial \rho(\xi,t)}{\partial t} + \frac{j(b,t,\rho(b,t)) - j(a,t,\rho(a,t))}{b-a} = 0. \tag{28}$$

After a Taylor expansion of the second expression, and in the limit $a \to b$, we obtain the differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} + \frac{\partial j}{\partial \rho}\frac{\partial \rho}{\partial x} = 0. \tag{29}$$

This is the conservation law in differential form, which mathematically is a (generally nonlinear) hyperbolic partial differential equation.

The following simple (but important!) example illustrates why one can not in general write

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \int_a^b \frac{\partial \rho(x,t)}{\partial t}dx. \tag{30}$$

A solid medium has a discontinuity in the density from ρ_1 to ρ_2. The medium moves with uniform speed U along the x-axis. Since the flux in this case is just ρU, the conservation law with the discontinuity *between* a and b, gives

$$\frac{d}{dt}\int_a^b \rho dx + \rho_2 U - \rho_1 U = 0, \tag{31}$$

or,

$$\frac{d}{dt}\int_a^b \rho dx = (\rho_1 - \rho_2)U. \tag{32}$$

However, since $\partial\rho/\partial t$ is equal to 0 except at the discontinuity, say at $x = x_g$, we could also write

$$\frac{d}{dt}\int_a^b \rho dx = \int_a^{x_g} \frac{\partial \rho_1}{\partial t}dx + \int_{x_g}^b \frac{\partial \rho_2}{\partial t}dx = 0 + 0 = 0. \tag{33}$$

Consequently,

$$\frac{d}{dt}\int_R \rho dx \neq \int_R \frac{\partial \rho}{\partial t}dx. \tag{34}$$

This is a situation which may well occur in practice.

Without getting too far into the theory of partial differential equations, we shall limit ourselves to a situation where $\mathbf{j}$ is a known, differentiable function of ρ. The differential equation is then reduced to

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \qquad c(\rho) = \frac{dj}{d\rho}. \tag{35}$$

The quantity $c(\rho)$ has dimension velocity, and is called the *kinematic velocity*. In a way which becomes more clear later, one could say that the kinematic speed represents the speed of information in the problem.

To find the solution of this differential equation, we make the following interesting observation: Suppose that the solution $\rho(x,t)$ is already known. Then we also know $c(\rho(x,t))$. Let us define a vector field in the (x,t)-plane by

$$\mathbf{v}(x,t) = \{c(\rho(x,t)), 1\} \tag{36}$$

Field curves (with the curve length parameter s) are defined by the equations

$$\begin{aligned} \frac{dx}{ds} &= c(\rho(x,t)), \\ \frac{dt}{ds} &= 1. \end{aligned} \tag{37}$$

Assume that $x = p(s)$ and $t = q(s)$, for $-\infty < s < \infty$ represent the curves. We calculate the variation of ρ along a curve by means of

$$\frac{d\rho}{ds} = \frac{\partial\rho}{\partial t}\frac{dt}{ds} + \frac{\partial\rho}{\partial x}\frac{dx}{ds} = \frac{\partial\rho}{\partial t}\cdot 1 + \frac{\partial\rho}{\partial x}\cdot c(\rho(x,t)) = 0. \tag{38}$$

This means that ρ is constant along the field curves. Consequently, $c = c(\rho)$ is also constant along the curves. But this implies in turn that the curves are straight lines. These fields lines are called *characteristic curves* or simply *characteristics*. Strictly speaking, the field lines are the projection in the (x,t)-plane of the real characteristics in (x,t,ρ)-space, but it is common also to call the projections for characteristics (In general, characteristic curves do not need to be straight lines).

We are now going to find the solution $\rho(x,t)$ for $-\infty < x < \infty$, $0 \leq t$, given that

$$\rho(x,0) = f(x). \tag{39}$$

With the condition given at $t = 0$, this is an initial value problem, also called the *Cauchy-problem* in this context.

However, if we are seeking the solution in a point (x_1, t_1), we first need to find the characteristic curve through the point. Since the characteristics are straight lines, they have equations

$$x = x_0 + c(\rho(x_0,0))\,t == x_0 + c(f(x_0))\,t \tag{40}$$

where $(x_0, 0)$ lies on x-axis. Accordingly, we must first find an x_0 such that

$$x_1 = x_0 + c(\rho(x_0,0))\,t_1. \tag{41}$$

Solving Eq. 41 implies solving an implicit (and in general nonlinear) equation in order to find x_0. Once we know x_0,

$$\rho(x_1,t_1) = \rho(x_0,0) = f(x_0), \tag{42}$$

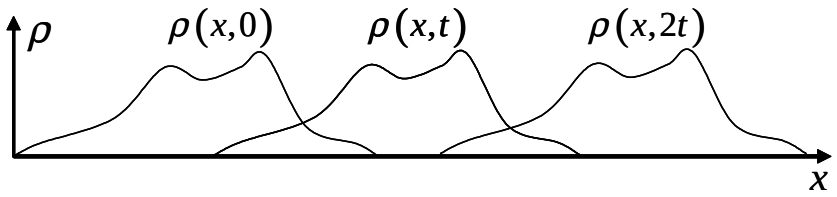

Figure 4: The solution of the simplest kinematic wave equation is a fixed function translated along the x-axis with constant speed.

since the value of ρ is constant along the characteristic. It is only possible to write the solution in explicit form in simple cases. Let us, as very a simple example, consider the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = 0. \tag{43}$$

Here, $c(\rho) \equiv 1$ so that the characteristics are the lines in (x,t)-plane defined by $x = x_0 + t$. Thus, we get $\rho(x,t) = \rho(x_0,0) = f(x_0) = f(x-t)$. The variation in the density at $t = 0$ thus moves without changing the shape towards the right with speed 1 as illustrated in Fig. 4.

2.5.1 The Riemann Problem

Let us again consider the 1D conservation law in integral form,

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx + j(\rho(b,t)) - j(\rho(a,t)) = 0, \tag{44}$$

with the differential formulation

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0,\ c(\rho) = \frac{dj}{d\rho}. \tag{45}$$

For this conservation law, there are three basic solutions that typically arise, and even if we can find this in most textbooks about Partial Differential Equations, e.g. the book of Whitham [18], we shall for completeness list them here as well. The three cases are solutions to the so-called *Riemann problem*, where we want to determine $\rho(x,t)$ for $-\infty < x < \infty$ and $0 \le t$ when

$$\rho(x,0) = \begin{cases} \rho_1, & x < 0 \\ \rho_2 & x > 0 \end{cases},\ \rho_1 \neq \rho_2. \tag{46}$$

The characteristics starting at x_0 on the x-axis, in this case, are given by

$$\begin{aligned} x &= x_0 + c(\rho_1)\,t, \quad x_0 < 0, \\ x &= x_0 + c(\rho_2)\,t, \quad x_0 > 0. \end{aligned} \tag{47}$$

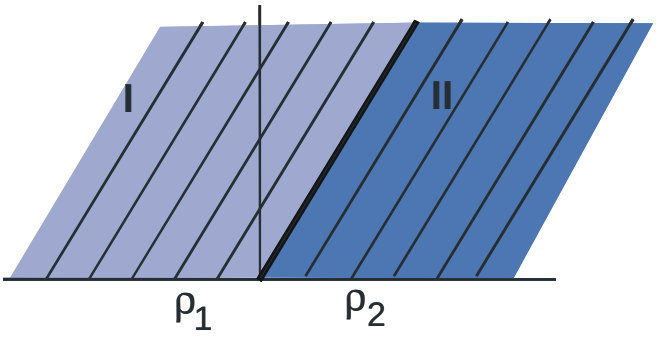

Figure 5: Contact discontinuity: The characteristics are parallel.

2.5.2 Contact Discontinuity

If we has $c(\rho_2) = c(\rho_1)$, the characteristics will be parallel for both positive and negative x_0. The region $-\infty < x < \infty$, $0 \leq t$, is now divided into two parts, I and II (see Fig. 5) where the solution ρ to the differential equation are, respectively, ρ_1 and ρ_2. The solution is called a *contact discontinuity*, since it is discontinuous along the contact of the two parts, i.e. the line $x = c(\rho_1)\, t = c(\rho_2)\, t$. Somewhat surprising, this solution does *not* need to be an acceptable solution for the conservation law. We shall see below that the conservation law is only satisfied if, in addition,

$$(\rho_1 - \rho_2)\, c(\rho_1) + j(\rho_2) - j(\rho_1) = 0 \tag{48}$$

and this does not need be the case even if $c(\rho_1) = j'(\rho_2) = c(\rho_2)$. If this extra condition

is not met, the solution of the conservation law develops in a more complicated way. The basic situation is however when the condition is fulfilled.

2.5.3 Rarefaction Wave

If $c(\rho_1) < c(\rho_2)$, the characteristics that start outside the origin have to go as shown in Fig. 6. The solution in regions I and III are thus ρ_1 and ρ_2, respectively.

If, $c(\rho)$ is *monotonically increasing* when ρ goes from $\rho_1$to ρ_2, the solution in region II becomes what is called an elementary *rarefaction wave*, *expansion wave*, or *expansion fan*. Here, all characteristics have to start at the origin. Therefore, the characteristics have all the equation $x = c(\rho)\, t$, and consequently, the solution for a point (x, t) in region II is given implicitly by

$$\rho(x, t) = c^{-1}(x/t) \tag{49}$$

(The inverse function c^{-1} exists under the above assumption of monotonicity). We leave to the reader to show that this solution really fulfil the conservation law.

Consider the following simple example where $c(\rho) = \rho$, and $\rho_1 = 0$ $\rho_2 = 1$. Then the characteristics for $x_0 < 0$ is simply $x = x_0$, whereas $x = x_0$ for $x_0 > 0$ 0 In region II defined

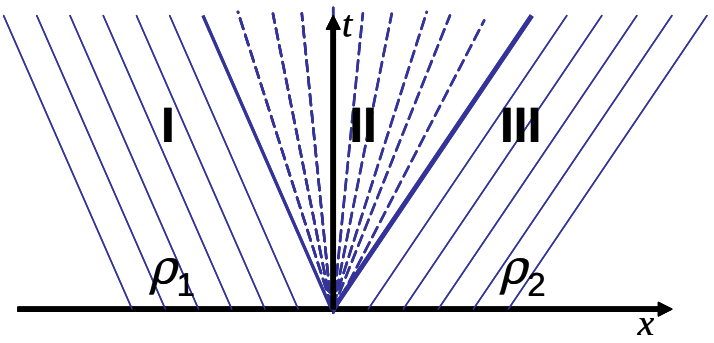

Figure 6: The rarefaction wave.

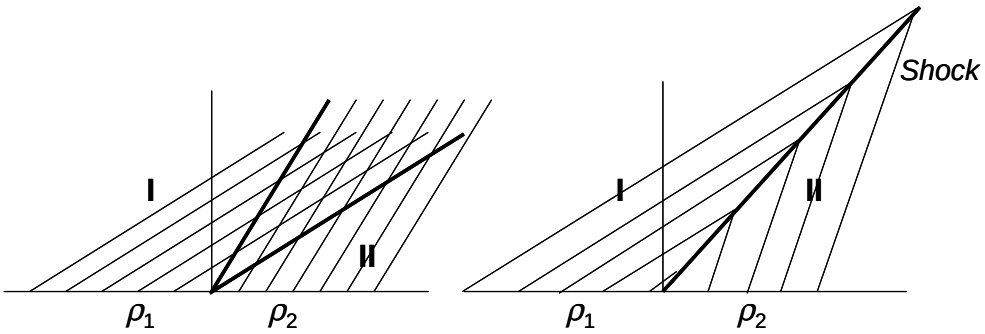

Figure 7: The situation when $c(\rho_1) > c(\rho_2)$ requires the introduction of a shock.

by $\{(x,t)\,; 0 < x < t\}$ the solution is given by $x = \rho t$, i.e. $\rho = x/t$. Thus, the solution for $t > 0$ becomes

$$\rho(x,t) = \left\{ \begin{array}{ll} 1, & x < 0, \\ x/t, & 0 \le x \le t, \\ 0, & t < x. \end{array} \right\} \tag{50}$$

Try some other possibilities for $c(\rho)$ and make sketches to see how things come out!

2.5.4 Shock Solution

If we have the reverse situation from above, namely that $c(\rho_1) > c(\rho_2)$, the characteristics will cross, as illustrated in Fig. 7 to the left. Although the solution outside the collision area can be found using the characteristic method, this is of little help in the area where the characteristics crosses. In order to resolve the situation, we have to go back to the original conservation law and introduce a *discontinuity,* $x = s(t)$, called a *shock*, as illustrated in Fig. 7 to the right.

In order to determine the speed $U = ds/dt$ of the shock, we consider interval $[a, b]$ such that it includes discontinuity. By calculating the change of the contents in $[a, b]$, as outlined

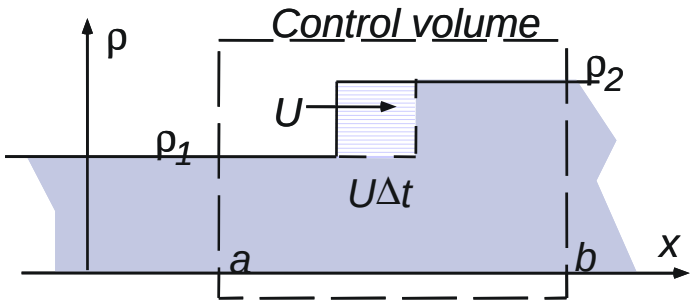

Figure 8: A control volume enclosing a discontinuity in the density. The change in the content during a time interval Δt is $(\rho_2 - \rho_1) U \Delta t$.

in Fig.8, we obtain

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dt = \lim_{\Delta t \to 0} \frac{-(\rho_2 - \rho_1) \times U\Delta t}{\Delta t} = (\rho_1 - \rho_2)\,U. \tag{51}$$

The conservation law will now only be satisfied if

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx + j\rho(b,t) - j(\rho(a,t)) = (\rho_1 - \rho_2)\,U + j(\rho_2) - j(\rho_1) = 0. \tag{52}$$

(recall the situation for the contact discontinuity). Thus we derive that the shock speed must be

$$U = \frac{j(\rho_2) - j(\rho_1)}{\rho_2 - \rho_1}, \tag{53}$$

and the solution for $t > 0$ is thus

$$\rho(x,t) = \left\{ \begin{array}{l} \rho_1, \quad x < Ut \\ \rho_2, \quad x > Ut \end{array} \right\} \tag{54}$$

We shall finally show a simple example where the same differential equation may come from different conservation laws. Let the density and the flux depend of another function u such that

$$\begin{aligned} \rho(x,t) &= \frac{1}{n} u(x,t)^n \\ j(x,t) &= \frac{1}{n+1} u(x,t)^{n+1} \end{aligned} \tag{55}$$

The differential formulation follows the conservation law $\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$,leading to

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = u^{n-1}\frac{\partial u}{\partial t} + u^n \frac{\partial u}{\partial x} = u^{n-1}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 0. \tag{56}$$

The differential equation for u is therefore essentially the same regardless the value of n we had the conservation law. The characteristics are the same in all cases, but the shock speeds are different for different n-s:

$$U = \frac{n}{n+1}\frac{u_2^{n+1} - u_1^{n+1}}{u_2^n - u_1^n} \tag{57}$$

The example shows that if the solutions develop shocks, the shock can *not* be found from the differential equation alone. The position of the shock must be determined from the original conservation laws. In some cases it is also necessary to bring in additional conditions in order to determine a physically acceptable solution (This is more thoroughly covered in courses in non-linear partial differential equations).

3 MODELING OF ROAD TRAFFIC

Systematic studies of road traffic started about 70 years ago in the homeland of the cars, USA. During the 1950s also mathematicians began to get stuck in rush hour traffic, and more theoretical work appears in mathematical journals. An article by M. J. Lighthill and G.B. Whitham in the Proceedings of the Royal Society entitled "*On kinematic waves II. A theory of traffic flow on long, crowded roads*" from 1955 is one of the milestones in the development. Mathematical modeling of road traffic is a relatively wide field, and there is much information on the Internet and in several textbooks.

The present material has been based on lectures and seminars in mathematical modeling at NTNU over several years. The report published by US Transportation Research Board, *Traffic Flow Theory* is, as of this writing, available free of charge from the Internet [7].

Research claims that between 20–30 percent of traffic jams on Norwegian roads would disappear if each motorist is driving more efficiently. Researchers at NTNU have in controlled trials managed to double the flux from 1800 to 3600 vehicles per hour just by adjusting the drivers' behavior. Interestingly enough, it appears that sometimes the flux of cars may be larger when the traffic is kept at 60km/h, compared to 90km/h. On the ring-road around London, *M25*, the speed limits are constantly adjusted in order to optimize the traffic flow.

Traffic modeling can be approached from many sides and applying many different mathematical and statistical tools. It is reasonable to think of models based of individual vehicles on a road where the speed is expressed as a function of road conditions and other vehicles nearby. The models may contain stochastic elements such as variations in the drivers' perception of what is a *safe speed*, a *safe distance to the vehicle in front,* and an *acceptable overtaking margin*. Such models quickly become analytically complicated, but are suitable for computer simulations. Queuing theory and other statistical models that describe the randomness of real traffic, are also widely used.

The article of Lighthill and Whitham suggests a continuum model for car traffic. Traffic flow is described in terms of density, flux, sources and sinks, which consequently leads to hyperbolic conservation laws. This theory is called the *kinematic theory* of road traffic.

The models can be refined by including the drivers' ability to respond to changes in traffic density, and how quickly they can adjust according to road conditions. The material below is mainly taken from the books of Whitham and Haberman stated in the reference list.

3.1 Kinematic Theory

In kinematic traffic theory, traffic is modeled by means of a simple conservation law. The density $\rho(x,t)$ of cars on the road is expressed as the number of cars per unit length. The term must of be considered somewhat pragmatic, as is often the case when we model a collection of highly discrete objects as a continuous medium. We consider ρ as a piecewise continuous function of position and time. Because of the car's finite size, it is reasonable to assume that

$$0 \leq \rho \leq \rho_{\max}, \tag{58}$$

where $\rho_{\max}$ is the maximum density as calculated from the cars' average length.

The car velocity v is assumed to be a function of car density, $v(\rho)$, so that $v(0) = v_{\max}$ and $v(\rho_{\max}) = 0$. Thus, v decreases as ρ increases. A clear-cut relation between the cars' velocity and density may only be reasonable on one-lane roads, but is also used for multi-lane highways, where the car velocities vary both individually and from lane to lane. In this case, one interprets v as the average speed, and measurements indicate this is a reasonable assumption, at least for parts of the interval between 0 and $\rho_{\max}$.

If we assume that the speed of the cars is a function of ρ, $v(\rho)$, a small argument gives us that the flux of cars, *i.e.* the number of cars passing a given point on the road per time unit, can be expressed as

$$J = \rho v(\rho). \tag{59}$$

In the traffic literature flux is often designated with the symbols k, or F, and the graph of J as a function of ρ is called the *fundamental diagram*. Normally, $J(\rho)$ tends to 0 when ρ approaches 0 or $\rho_{\max}$, and is a concave function with a maximum value somewhere between 0 and $\rho_{\max}$. The conservation law becomes as before

$$\frac{d}{dt}\int_a^b \rho(x,t)dx + J(b,t) - J(a,t) = \int_a^b q(x,t)dx, \tag{60}$$

where the source term expresses cars entering or leaving the road. The differential formulation leads to a first order hyperbolic equation, and since J is only a function of ρ, we can write the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{dJ}{d\rho}\frac{\partial \rho}{\partial x} = q. \tag{61}$$

If $q = 0$, the characteristics will be straight lines with slope (kinematic velocity) equal to $c(\rho) = \frac{dJ}{d\rho}$.

By applying the equations 60 and 61, we can examine what is happening around a traffic light crossing, when we have a varying density of traffic, how the individual cars are moving, etc. The car's own motion is determined by differential equation

$$\frac{dx}{dt} = v(\rho(x,t)). \tag{62}$$

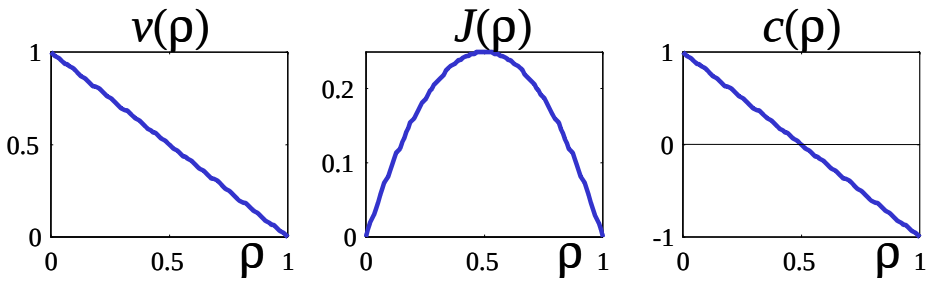

Figure 9: Speed, flux and kinematic velocity as a function of car density for the standard model.

Usually, $c(\rho)$ will be a decreasing function of ρ. This will typically lead to situations developing shocks if the traffic is moving in the positive x-direction and ρ increases with x. From the conservation law, a shock $x = s(t)$ will have to satisfy

$$\frac{ds}{dt} = \frac{J(s+,t) - J(s-,t)}{\rho(s+,t) - \rho(s-,t)}, \tag{63}$$

Changes in traffic conditions can be incorporated in several ways. If the road has a narrowing, e.g., goes from two to one lane, it is reasonable that $\rho_{\max}$ is reduced, whereas $v_{\max}$ remains the same. This changes the fundamental diagram. At the start of the narrowing, the flux has to be continuous, whereas ρ will have a discontinuity.

If the road is slippery and the visibility is poor due to rain or fog, then $v_{\max}$ will decrease whereas $\rho_{\max}$ remains unchanged. This changes the fundamental diagram in a different way.

A very common kinematic model that is reasonably easy to work with analytically, is to assume that v is a decreasing, linear function of ρ. After scaling, we obtain the equations

$$\begin{aligned} v(\rho) &= 1 - \rho, \\ J(\rho) &= \rho(1-\rho), \\ c(\rho) &= 1 - 2\rho, \\ \frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} &= 0. \end{aligned} \tag{64}$$

Fig. 9 shows how the car velocity, the flux and the kinematic velocity change for the this model. Note that $c(\rho) = \frac{d}{d\rho}(\rho v) = v + \rho v'(\rho)$, and therefore $c(\rho) \neq v(\rho)$ when $\rho \neq 0$. Below we shall use this model to analyze some simple situations.

3.1.1 Traffic Lights

Assume that at $x = 0$, there has a been a red light for the cars for $t < 0$. To the left of the light ($x < 0$), there is a dense queue of cars, $\rho = 1$, while at the the right ($x > 0$), there are

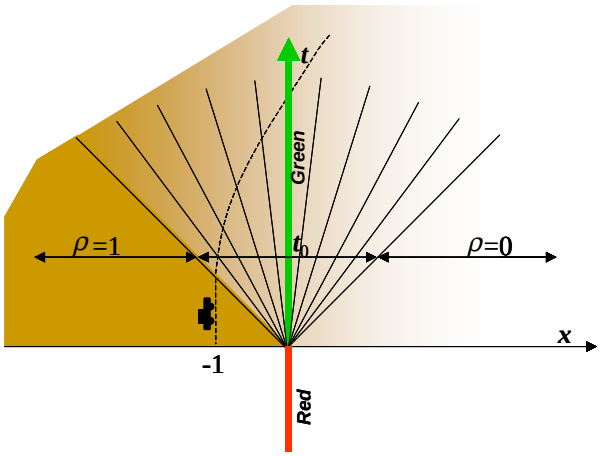

Figure 10: Junction at $x = 0$, where the light changes from red to green.

no cars and $\rho = 0$. In all such problems, it is useful to outline a so-called x/t-chart that describes the conditions, in particular how the characteristics behave, as shown in Fig. 10. When $t > 0$, we have three regions. To the left is an area where $\rho = 1$, to the right a region where $\rho = 0$, while in the middle there is an expansion wave with characteristics starting at the origin,

$$x = c(\rho)\, t = (1 - 2\rho)\, t. \tag{65}$$

Within the central area $\rho = \frac{1}{2}\left(1 - \frac{x}{t}\right)$, and the complete solution becomes for $t = t_0$

$$\rho(x, t_0) = \begin{cases} 1, & x \le t_0, \\ \frac{1}{2}\left(1 - \frac{x}{t_0}\right) & -t_0 < x < t_0, \\ 0 & x \ge t_0. \end{cases} \tag{66}$$

Suppose you are sitting in a car at $x = -1$ for $t = 0$. What will be your own motion for $t > 0$? From Fig. 10 we observe that you will start to drive at $t = 1$, and then your own path, $y(t)$, will be controlled by the differential equation

$$\begin{aligned} \frac{dy}{dt} &= v(\rho) = 1 - \frac{1}{2}\left(1 - \frac{y}{t}\right) \\ &= \frac{1}{2}\left(1 + \frac{y}{t}\right), \; y(1) = -1. \end{aligned} \tag{67}$$

The equation is thus $2t\dot{y} - y = t$, with general solution $y(t) = At^{1/2} + t$. Since $y(1) = -1$, $y(t) = t - 2t^{1/2}$. It is worth noting that the distance to the front car at $x = t$ becomes greater as time passes.

The situation we have analyzed, resembles what you encounter in a big running event: if you have ambitions of fighting in the lead, it pays to have a position as close as possible to the head of the queue before the start.

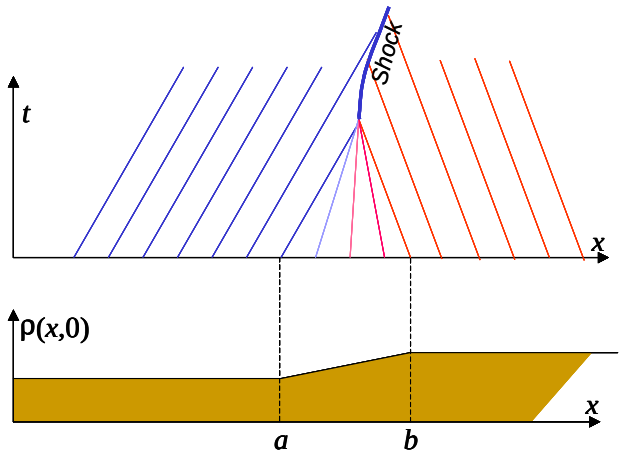

Figure 11: The situation with an increase in the car density.

3.1.2 Traffic Clogging Up

Assume that there is a line of traffic on the road where the density at $t = 0$ has the form

$$\rho(x,0) = \begin{cases} \rho_1, & x \le a, \\ \rho_1 + \frac{x-a}{b-a}(\rho_2 - \rho_1) & a < x < b, \\ \rho_2 & b \le x. \end{cases} \tag{68}$$

where $\rho_1 < \rho_2$. Between $x = a$ and $x = b$, the car density increases linearly from ρ_1 to ρ_2. The characteristics are given by $x = x_0 + (1 - 2\rho)\,t$, and for characteristics between a and b this amounts to

$$\begin{aligned} x &= x_0 + t - 2\left(\rho_1 + \frac{x_0 - a}{b-a}(\rho_2 - \rho_1)\right)t \\ &= x_0 + (1 - 2\rho_1)\,t - 2\frac{x_0 - a}{b-a}(\rho_2 - \rho_1)\,t. \end{aligned} \tag{69}$$

By inserting the time $t_s = \frac{1}{2}\frac{b-a}{\rho_2-\rho_1}$, we see that x_0 vanishes. This means that all characteristics starting from the interval $[a, b]$ meet in the point (x_s, t_s),

$$x_s = a + (1 - 2\rho_1)\frac{1}{2}\frac{b-a}{\rho_2 - \rho_1} = b + (1 - 2\rho_2)\frac{1}{2}\frac{b-a}{\rho_2 - \rho_1}. \tag{70}$$

The situation is sketched in Fig. 11.

For $t > t_s$, we get a jump in density, a *shock*. The speed of the shock, has to be determined from the shock condition, as discussed above,

$$\begin{aligned} U &= \frac{J(\rho_2) - J(\rho_1)}{\rho_2 - \rho_1} \\ &= \frac{\rho_2(1-\rho_2) - \rho_1(1-\rho_1)}{\rho_2 - \rho_1} \\ &= 1 - \rho_1 - \rho_2. \end{aligned} \tag{71}$$

Region	ρ	$v(\rho)$	$J(\rho)$	$c(\rho)$
$x<1$	$[0\ 1]$	$1-\rho$	$\rho(1-\rho)$	$1-2\rho$
$x>1$	$[0\ \frac{1}{2}]$	$1-2\rho$	$\rho(1-2\rho)$	$1-4\rho$

Table 1: The conditions surrounding a narrowing of the road..

One may wonder what is happening around such a shock, and in practice, the cars will try to avoid colliding. However, when driving through the shock, the car velocity has a discontinuity, and it is limited how fast it is possible to react!

3.1.3 When is the First Shock Formed?

The shock is formed when two characteristics collide. Let us look at two characteristics starting at x_0 and $x_0+\Delta t$, respectively, and meeting at (x_s,t_s):

$$x_0+c(x_0)t_s = x_0+\Delta t+c(x_0+\Delta t)t_s. \tag{72}$$

Thus,

$$t_s = -\frac{1}{\left(c(x_0+\Delta t)-c(x_0)\right)/\Delta t}. \tag{73}$$

If we let $\Delta t\to 0$, the limit is

$$t_s = -\frac{1}{\frac{dc}{dx}\big|_{x_0}}.$$

For a shock to form for $t>0$, there has to be some x_0 where $\frac{dc}{dx}\big|_{x_0}<0$, and the first time this happens is

$$\min t_s = -\frac{1}{\min_x \frac{dc}{dx}}. \tag{74}$$

3.1.4 Narrowing the Road

On a road with two lanes for $x<1$, one of the lanes is closed for $x>1$, so that the maximum car density for $x>1$ is only half of the original. Table 1 shows how the flux varies with the density of each of the two parts of the road. There is no storage for cars at $x=1$. Therefore, the flux around $x=1$ must be continuous,

$$J\left(\rho(1-,t)\right) = J\left(\rho(1+,t)\right). \tag{75}$$

This means that the density is discontinuous (if different from 0). If the number of vehicles passing $x=1$ is as large as possible, the flux is $J=1/8$, and densities immediately to the left and right of $x=1$ are given by

$$J\left(\rho^-\right) = \rho^-\left(1-\rho^-\right) = J\left(\rho^+\right) = \rho^+\left(1-2\rho^+\right). \tag{76}$$

Thus,

$$\rho^+ = 1/4, \tag{77}$$

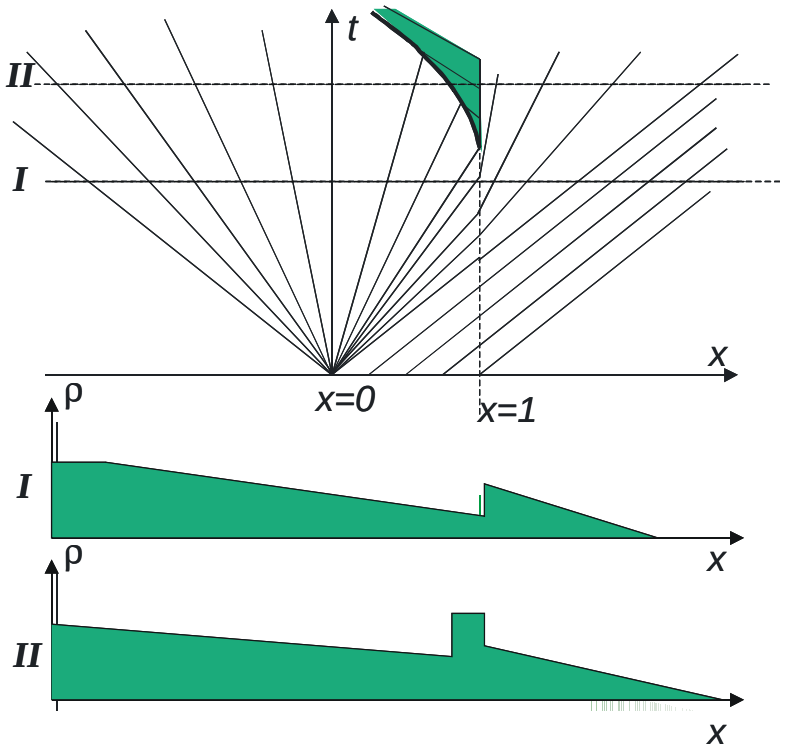

Figure 12: Narrowing of the road after a traffic light.

whereas there are two possibilities for ρ^-:

$$\rho^- = \frac{1}{2}\left(1 \pm \frac{\sqrt{2}}{2}\right). \tag{78}$$

To see how both possibilities can occur, we connect this situation with a traffic light at $x = 0$, as in the first example. The situation is illustrated in Fig. 12.

We get a shock at $x = 1$ until the density ρ on the left side reaches the value $\rho_1^- = \frac{1}{2}\left(1 - 2^{-1/2}\right)$. Then we get a sudden jump in the density up to $\rho_2^- = \frac{1}{2}\left(1 + 2^{-1/2}\right)$. Before the narrowing, we get a queue where the density is ρ_2^-. At the end of this queue another shock is formed. We leave it to readers to consider how it is to drive through such a situation.

3.1.5 Research Project: *A Green Wave in Infinity Street?*

In a long straight street the pedestrian crossings are organizes\d with traffic lights. If the traffic lights are uncoordinated, the cars will need to drive and stop at uneven interval, and the resulting average flux of cars may be quite low. However, sometimes we hear that it is possible arrange the lights in a so-called green wave, so that the cars may "surf" through the street without having to stop. Is it really possible to streamline the traffic by using

the green waves, and what is the maximum possible average flux? In this study we will consider an idealized situation of this problem.

At *Infinity Street* pedestrian crossings are located at a constant distance L. There are no side streets with opportunities for for the cars to leave or enter the street. The crossings have all traffic lights with a cycle of length S. This means that it is *red* in the period $[0, r^*)$, then it is *green* in the period $[r^*, S)$. The cycle repeats itself continuously. To facilitate the driving, the light cycles may be displaced in time in relation to each other, so that cycle of crossing $k+1$ starts some time before or after the cycle of crossing k. The maximum vehicles density is $\rho_{\max}$, and the velocity is $v_{\max}$. Traffic follows the simple kinematic model considered above. By scaling x and ρ the usual way, and using the time scale $L/v_{\max}$ we obtain Eq. 64.

The length of the red period is denoted r and green period g. Here $r = 1$ correspond to the shortest time it takes to drive between two traffic lights separated by a length 1, and the cycle length is $r+g$. The traffic goes around the clock, and the problem is to maximize the average flux, $\bar{J}$. If the red period starts in $x = 0$ at $t = 0$, the average flux will be

$$\bar{J} = \frac{1}{r+g}\int_0^{r+g} J(0,t)dt = \frac{1}{r+g}\int_r^{r+g} \rho(0,t)[1-\rho(0,t)]dt. \tag{79}$$

It is clear that the theoretical maximum average flux will be

$$\bar{J}_{\max} = \frac{g}{r+g}J_{\max} = \frac{g}{r+g}\frac{1}{4}, \tag{80}$$

since the maximum number of cars passes the crossing during the green period, and anything better than that is impossible.

We shall first analyze a simpler situation with one traffic light. Until time $t = -g$ there has been red light at $x = 0$. To the left of $x = 0$ there is a queue of cars. The light is green up to $t = 0$, where it again changes to red. From the point $(0, -g)$ there is now an expansion wave which collides with the shock from $(0, 0)$ in both positive and negative x-directions, as shown in Fig. 13. The shock speed in this case is

$$U = \frac{J_2 - J_1}{\rho_2 - \rho_1} = 1 - (\rho_1 + \rho_2). \tag{81}$$

The density of the two characteristics that are symmetrical about the t-axis are $\rho^- = \frac{1}{2} - \rho$ and $\rho^+ = \frac{1}{2} + \sigma$, respectively. Thus, the two shock speeds the same with the opposite sign:

$$\begin{aligned} U^+ &= 1 - (0 + \frac{1}{2} - \sigma) = \frac{1}{2} + \sigma, \\ U^- &= 1 - (1 + \frac{1}{2} + \sigma) = -\left(\frac{1}{2} + \sigma\right). \end{aligned} \tag{82}$$

Consequently, the two shock curves are also symmetrical about the origin. We can calculate

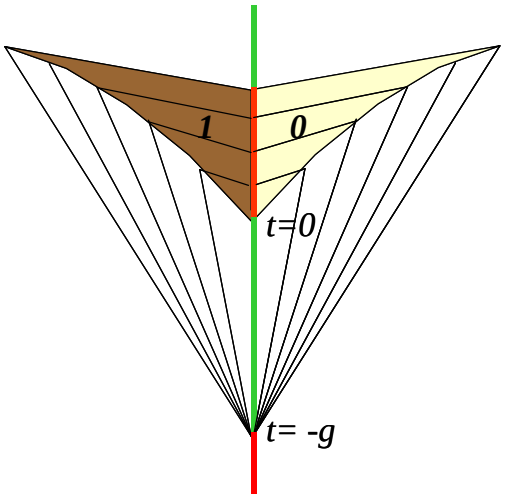

Figure 13: The expansion wave from $(0, -g)$ collides with the shocks that start at $t = 0$. The characteristics in the regions where $\rho = 0$ and 1 have speed 1 and -1, respectively.

the shock curve x_c from

$$\begin{aligned} \frac{dx_s}{dt} &= U^+(x_s, t) = 1 - \rho^+ \\ &= 1 - c^{-1}\left(\frac{x_s}{t+g}\right) \\ &= 1 - \frac{1}{2}\left(1 - \frac{x_s}{t+g}\right), \end{aligned} \tag{83}$$

or

$$2(t+g)\dot{x}_s = (t+g) + x_s. \tag{84}$$

The solution is

$$x_s(t) = t + g - g^{1/2}(t+g)^{1/2}. \tag{85}$$

The shocks follow this curve until they meet the characteristics that have started in $t = r$, in other words, immediately after the red period from $t = 0$ to r is over. The areas with $\rho = 0$ and $\rho = 1$ are therefore also *symmetric* about the y-axis.

Let us now look at a situation where the red period from $x = 0$ has lasted so long that the areas in Fig. 13 reaches out to $x = \pm\frac{1}{2}$. We find out where $(\pm\frac{1}{2}, t_0)$ is by observing that

$$\frac{1}{2} = 0 + c(0)(t_0 - r),$$

that is,

$$t_0 = r + \frac{1}{2}\frac{1}{c(\rho = 0)} = r + \frac{1}{2}. \tag{86}$$

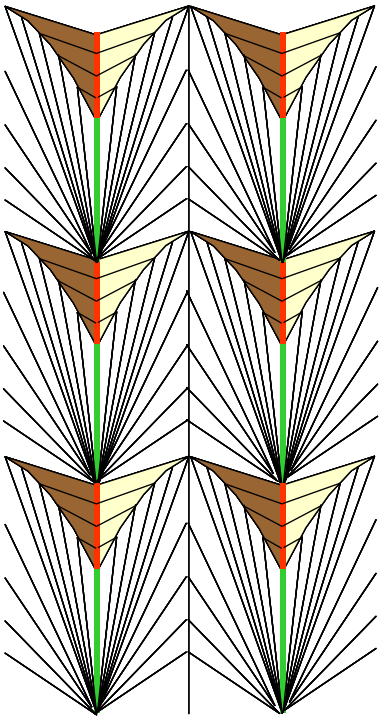

Figure 14: Sketch of the complete solution. Note that we get a stationary shock at $x = 1/2$.

Then we put this into the equation for the shock:

$$\begin{aligned}\frac{1}{2} &= (t_0 + g) - g^{1/2}\,(t_0 + g)^{1/2} \\ &= \left((r + \frac{1}{2}) + g\right) - g^{1/2}\left((r + \frac{1}{2}) + g\right)^{1/2},\end{aligned} \tag{87}$$

or

$$g = \frac{r^2}{\frac{1}{2} - r}. \tag{88}$$

The solution requires that $r < 1/2$, and otherwise that $g = r$ for $r = 1/4$. For these combinations of g and r, we can now construct complete solutions for *Infinity Street* that for any given ratio $r/g \in \mathbb{R}^+$ provide maximum throughput. The construction is best described on a figure, see Fig. 14. Note that we have got a motionless shock for half integers (the shock speed is 0, since the sum of the densities on both sides is 1 at any time). If we want r/g to be large, this gives very short cycles. Even the maximum red period, $r = 1/2$, which corresponds to the time it takes to cover half the distance between two traffic lights at maximum speed, seems to be rather short for practical purposes.

It appears that the symmetric structure may be generalized:

- At shorter distances between the crossings, the "leaves" are cut and meet in the middle.

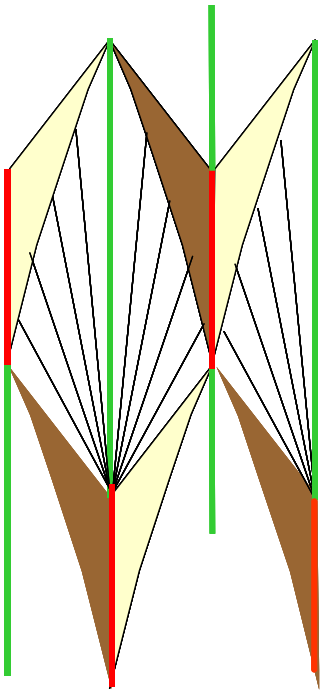

Figure 15: Is this a possible solution?

- For larger distances the "leaves" are extended with two shocks that also meet in the middle.

Is it possible to have a solution that is looking like the one in Fig. 15?

3.2 Generalizations of the Kinetic Theory

In practice, drivers try to compensate for changes in traffic density by adjusting the speed according to surrounding conditions. To avoid the development of shocks and thus extremely rapid changes in density, they will tend to slow down somewhat more than the relationship $v = v(\rho)$ suggests, and therefore avoid strong gradients in density build up. Whitham models this by assuming that the drivers adjust the speed as

$$v = v_k(\rho) - \kappa \frac{\partial \rho / \partial x}{\rho}, \tag{89}$$

where v_k is the speed according to the kinematic theory above. This seems to be a reasonable model since $v = v_k(\rho)$ if ρ is constant, whereas v is less than v_k if ρ increases in the driving direction, and the opposite when ρ decreases. The best part of the model, and probably the reason why Whitham selected the particular form in Eq. 89, is that it leads to a famous 2nd order parabolic equation for ρ, since $J = \rho v = \rho v_k(\rho) - \kappa \rho_x$:

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} - \kappa \frac{\partial^2 \rho}{\partial x^2} = 0, \; c(\rho) = \frac{d(\rho v_k)}{d\rho}. \tag{90}$$

For the special case where $c(\rho) = 1-2\rho$ we may use c instead of ρ as the dependent variable, leading to the equation

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} - \kappa\frac{\partial^2 c}{\partial x^2} = 0, \tag{91}$$

called *Burgers Equation*. Burgers equation is one of the most studied non-linear partial differential equations and is extensively described in, e.g. [18], Ch. 4.

It turns out that Burgers equation has solutions in the form of "migrating fronts":

$$\begin{gathered} c(x,t) = C(x - x_0 - at) = C(s),\ s = x - x_0 - at. \\ \lim_{s\to-\infty} C(s) = c_1,\ \lim_{s\to\infty} C(s) = c_2. \end{gathered} \tag{92}$$

The front thus has a constant value on the curves $x = x_1 + at$, and well-defined limits as x tend to $\pm\infty$. We find such a solution by entering C in the equation:

$$-aC' + CC' - \kappa C'' = 0. \tag{93}$$

By one integration we obtain

$$(C-a)^2 = 2\kappa C' + A, \tag{94}$$

where A is another integration constant. Since we expect that also the derivative goes to 0 when $x \to \pm\infty$, we see that

$$\begin{aligned} a &= \frac{c_1 + c_2}{2}, \\ A &= \left(\frac{c_1 - c_2}{2}\right)^2. \end{aligned} \tag{95}$$

We now insert this and separate the variables:

$$-\frac{ds}{2\kappa} = \frac{dC}{\left(\frac{c_2-c_1}{2}\right)^2 - (C-a)^2}. \tag{96}$$

This gives

$$-\frac{s - s_0}{2\kappa} = \frac{2}{c_2 - c_1}\operatorname{arctanh}\left(2\frac{C-a}{c_2-c_1}\right), \tag{97}$$

which may be turned around to

$$\begin{aligned} C(s) &= \frac{c_1+c_2}{2} - \frac{c_2-c_1}{2}\tanh\left(\frac{c_2-c_1}{4\kappa}(s-s_0)\right) \\ &= c_1 + (c_2-c_1)\left[1 + \exp\left(\frac{c_2-c_1}{2\kappa}(s-s_0)\right)\right]^{-1}. \end{aligned} \tag{98}$$

The function is a *sigmoid* (recall the equation for logistic growth). If $c_2 < c_1$, then $C(s)$ is a monotone decreasing front moving at velocity

$$U = a = \frac{c_1 + c_2}{2} \tag{99}$$

from to left to right. Note that since $c = 1 - 2\rho$ in this model, $\rho(x,t)$ will be a monotone *increasing* front moving with the same velocity. Moreover,

$$\begin{aligned} \lim_{x\to-\infty} \rho(x,t) = \rho_1 = \frac{1-c_1}{2}, \\ \lim_{x\to\infty} \rho(x,t) = \rho_2 = \frac{1-c_2}{2}, \\ U = \frac{c_1+c_2}{2} = 1 - (\rho_1+\rho_2). \end{aligned} \tag{100}$$

The transition between $\rho = \rho_1$ and $\rho = \rho_2$ becomes sharper the smaller κ is, but U is independent of κ. We see further that U is just the shock speed for the equation

$$\rho_t + \rho\rho_x = 0 \tag{101}$$

for the initial conditions $\rho(x,0) = \rho_1$ for $x < 0$, and $\rho(x,0) = \rho_2$ for $x > 0$, and $\rho_1 < \rho_2$. The behaviour in the limit $\kappa \to 0$ is therefore reasonable.

Whitham take the theory a step further by studying what happens when the cars need some time to adjust to the proper speed. A driver has some finite reaction time. If we look at the speed of a particular car, v_B, in a situation where traffic density changes, the car velocity could probably follow the equation

$$\frac{dv_B}{dt} = \frac{1}{\tau}\left(V - v_B\right), \tag{102}$$

where V is the ideal speed of the drivers wants to follow, as given by Eq. 102. This is a well-known equation from control theory. If $v_B = v_0$ at time $t = 0$ and V is a given function of t from 0 to t, it is possible to express v_B as

$$v_B(t) = v_0 e^{-t/\tau} + \int_0^t \exp(\frac{t'-t}{\tau}) V(t')dt'. \tag{103}$$

If $V = V_0$ is constant for $t > 0$, the solution becomes

$$v_B(t) = (v_0 - V_0)e^{-t/\tau} + V_0. \tag{104}$$

The effect of the initial value will die out with time constant τ, and if V varies slowly (that is, the time scale for variations in V is longer than τ), v_B will tend to follow V.

We want to formulate Eq. 102 for the velocity v as a function of both t and x. The car follows a path $x(t)$, so that $v_B(t) = v(x(t),t))$. This requires what in mechanics is called the convective derivative,

$$\begin{aligned} \frac{dv_B}{dt} &= \frac{d}{dt}v(x(t),t) \\ &= \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial t} \\ &= \frac{\partial v}{\partial x}v + \frac{\partial v}{\partial t}. \end{aligned} \tag{105}$$

Altogether, this gives *two* coupled equations for the motion

$$\begin{aligned}\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}&=0,\\ \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}&=\frac{1}{\tau}\left(\left(v_k(\rho)-\kappa\frac{\partial\rho/\partial x}{\rho}\right)-v\right).\end{aligned}\tag{106}$$

The equations are called a *hyperbolic system*, and we shall not go into the solution them in full detail, but we see immediately that the system has *equilibrium solutions*:

$$\begin{aligned}\rho&=\rho_0,\\ v_0&=v_k(\rho_0).\end{aligned}\tag{107}$$

The way of investigating whether the equilibria are stable or unstable is to write

$$\begin{aligned}\rho&=\rho_0+r,\\ v&=v_0+w,\end{aligned}\tag{108}$$

where $r\left(x,t\right)$ and $w\left(x,t\right)$ are small perturbations. After some arithmetic, we arrive at the following system of equations, where we have inserted $c_0=\frac{d}{d\rho}(v_k\rho)|_{\rho=\rho_0}=v_k'(\rho_0)\rho_0+v_0$:

$$\begin{aligned}r_t+v_0r_x+\rho_0w_x&=0,\\ w_t+v_0w_x&=-\frac{1}{\tau}\left(w-\frac{c_0-v_0}{\rho_0}r+\frac{\kappa}{\rho_0}r_x\right).\end{aligned}\tag{109}$$

The equations may be combined into a single linear 2nd order equation containing only r:

$$r_t+c_0r_x=\kappa r_{xx}-\tau\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial x}\right)^2r.\tag{110}$$

We may recognize the first part,

$$r_t+c_0r_x=\kappa r_{xx},$$

as a linear convection/diffusion equation, which is known to have solutions quickly dying out with time. However, the last term, $-\tau\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial x}\right)^2r$, may create problems for us if it "dominates" over the diffusion term κr_{xx}. The standard method for studying such linear equations is to examine *Fourier components*:

$$r(x,t)=ae^{i(kx-\omega t)}.\tag{111}$$

The Fourier component is a travelling wave with wave number k and frequency ω. If this is inserted into the equation, we get a so-called *dispersion relation* linking k and ω:

$$\tau(\omega-v_0k)^2+i(\omega-c_0k)-\kappa k^2=0.\tag{112}$$

Since the wave number $k=2\pi/\lambda$, where λ is the wave length, k is a real number. On the other hand, the frequency will generally be complex. Moreover, since

$$ae^{i(kx-\omega t)}=ae^{(\operatorname{Im}\omega)t}e^{i(kx-(\operatorname{Re}\omega)t)},\tag{113}$$

we see that if the imaginary part of ω is greater than 0, the amplitude of the Fourier component will grow exponentially in time, while is decreases exponentially if $\operatorname{Im}\omega < 0$. It is possible to show that

$$\operatorname{Im}\sqrt{\frac{\kappa}{\tau}} < c_0 < v_0 + \sqrt{\frac{\kappa}{\tau}}. \tag{114}$$

This tells us that v_0 and c_0 can not be too different. For all Fourier components to die out we must have

$$|v_0 - c_0| < \sqrt{\frac{\kappa}{\tau}}. \tag{115}$$

If we introduce $v_k(\rho) = 1 - \rho$, we see that $v_0 = 1 - \rho_0$ and $c_0 = 1 - 2\rho_0$ (in dimensionless variables). Therefore, the perturbations above die out only when

$$\rho < \sqrt{\frac{\kappa}{\tau}}. \tag{116}$$

This result, which does not seem to be mentioned in Whitham, is interesting in the light of observations that have been reported. The traffic seems to follow a fundamental diagram from $\rho = 0$ and up to a certain ρ_c which is considerably less than $\rho_{\max}$. For larger densities the well-defined behaviour breaks down, and J are significantly smaller than the fundamental diagram would indicate. It could therefore be interesting to know whether this is due to instabilities of the type we found here, but the problem is to find realistic numerical values to insert for τ and κ.

3.3 Individual Car Models

The theory in this section is mostly obtained from the book by Haberman, Sec. 64 [9].

Individual car models deal with individual cars on the road. With one lane, and no possibility of passing, we can assume that each car adjusts its own speed relative to the speed of the car ahead (assuming it is so close that the driver can see it). Let us denote the position of the vehicle No. n in the queue by $x_n(t)$.

$$\frac{d^2x_n}{dt^2} = -\lambda\left(\frac{dx_n}{dt} - \frac{dx_{n-1}}{dt}\right). \tag{117}$$

If car number n has higher speed than the vehicle in front (number $n-1$), car number n will brake. Actually, there will be some delay in the reaction of drivers, so we should write

$$\frac{d^2x_n(t+T)}{dt^2} = -\lambda\left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}\right), \tag{118}$$

where T expresses the drivers' response time. The equation may be integrated once:

$$\frac{dx_n(t+T)}{dt} = -\lambda\left(x_n(t) - x_{n-1}(t)\right) + d_n. \tag{119}$$

In a uniform situation where all cars have the same speed and same distance, the density will be

$$\rho = \frac{1}{x_{n-1} - x_n}. \tag{120}$$

In that case, the cars' velocity is thus

$$v = \frac{dx_n}{dt} = -\frac{\lambda}{-\rho} + d. \tag{121}$$

The constant d is chosen so that $v = 0$ for $\rho = \rho_{\max}$, and this leads to the following expression for v and the flux J:

$$v = \lambda \left(\frac{1}{\rho} - \frac{1}{\rho_{\max}} \right), \tag{122}$$

$$J = v\rho = \lambda\rho \left(\frac{1}{\rho} - \frac{1}{\rho_{\max}} \right). \tag{123}$$

The model is called the *California model*. The model is not very realistic for small densities since $v \to \infty$ when $\rho \to 0$. If there are very few cars on the road, where will be no car in sight most of the time, and then it is reasonable to move at maximum speed. A modified model would be to set

$$v = \min \left(v_{\max}, \lambda \left(\frac{1}{\rho} - \frac{1}{\rho_{\max}} \right) \right). \tag{124}$$

It is also possible to embed the drivers sensitivity for changes by assuming that λ varies with the distance to the car in front. We could for example assume that

$$\lambda = \frac{a}{x_{n-1} - x_n},$$

which means that the sensitivity disappears when the distance is great. The Eq. (117) then modifies to

$$\frac{d^2 x_n}{dt^2} = -\frac{a}{x_{n-1} - x_n} \left(\frac{dx_n}{dt} - \frac{dx_{n-1}}{dt} \right),$$

which can still be integrated analytically to

$$\frac{dx_n(t+T)}{dt} = -a \ln |x_n(t) - x_{n-1}(t)| + d_n. \tag{125}$$

A similar argument to the above then yields

$$v = a \ln \frac{\rho_{\max}}{\rho}, \tag{126}$$

$$J = \rho a \ln \frac{\rho_{\max}}{\rho}. \tag{127}$$

This still causes $v \to \infty$ when $\rho \to 0$, but here $J \to 0$ for both $\rho = 0$ and $\rho = \rho_{\max}$. The model is called the *Greenberg model*, and, according to [18], it fits the traffic in the Lincoln Tunnel from Manhattan to New Jersey when

$$\begin{aligned} a &= 17.2 \text{ mph}, \\ \rho_{\max} &= 228 \text{ cars/mile}. \end{aligned} \tag{128}$$

The model is still not entirely satisfactory, but the same idea could be taken further, *e.g.* assuming that

$$\lambda = \frac{\tilde{a}}{(x_{n-1} - x_n)^2}. \tag{129}$$

We then get a linear relationship between v and ρ. Measurements of flux as a function of density suggest a finite derivative at $\rho = 0$, in line with v going to a finite value when $\rho \to 0$.

3.3.1 Instabilities in a Queue

Driving in a queue, we may have experienced that it is uncomfortable to be behind drivers who keep an irregular speed, and this may also be analyzed with individual car models. With a finite reaction time, the equations are no longer pure differential equations, but what is denoted *Delay Equations*. Consider two cars and assume that the speed $v_1$of the car in front vary periodically. It turns out to be convenient to work with complex solutions, but the result using physical real periodic solutions will be the same. We therefore assume that

$$v_1(t) = 1 + ae^{i\omega t}, \tag{130}$$

where the amplitude a is much smaller than 1. Assume that we have reached a stationary situation where the following car has a similar variation in the velocity,

$$v_2(t) = 1 + be^{i\omega t}. \tag{131}$$

We put this into Eq. (119) and get

$$bi\omega e^{i\omega T} = -\lambda(b - a), \tag{132}$$

or

$$b = \frac{1}{1 + \frac{i\omega}{\lambda} e^{i\omega T}} a. \tag{133}$$

Thus

$$\frac{|b|}{|a|} = \left| \frac{1}{1 + \frac{i\omega}{\lambda} e^{i\omega T}} \right| = \frac{1}{\sqrt{\left(1 - 2\frac{\omega}{\lambda} \sin \omega T + \left(\frac{\omega}{\lambda}\right)^2\right)}}. \tag{134}$$

The amplitude of b will be greater than the amplitude of a if the denominator is less than 1 This occurs when

$$\sin(\omega T) > \frac{\omega}{2\lambda}, \tag{135}$$

which can be expressed as

$$\frac{\sin \omega T}{\omega T} > \frac{1}{2\lambda T}. \tag{136}$$

Since $\sin x/x \leq 1$, there is no danger as long as $\lambda T < 1/2$, but if this is not the case, there are low frequencies where the amplitude for the second car is larger than for the car in front. With several cars in the line, a further magnification will occur for cars further back. Good drivers will notice this and try to dampen the fluctuations in the speed. A similar analysis could also be considered for a sudden braking of the first car.

4 CONSERVATION LAWS OF MECHANICS

It turns out that the physics of the continuum matter surrounding us, *i.e.* solid material, liquids and gases, can be described in a compact way using the framework above, and this is acknowledged in most recent books about continuum and fluid mechanics, for example [15].

However, the laws of physics are basically laws for a given collection of matter. For example, Newton's laws are laws for one or more "mass points". In the same way, a thermodynamic system, as we consider it when formulating the first law of thermodynamics, consists of a fixed collection of molecules. In continua like liquids and gases, where the material is moving, we are mostly interested in formulating the laws for a fixed region of space, that is, a *control volume*. The control volume will therefore contain different mass particles at different times. .

In mechanics, a continuous medium in motion can be described in the *Eulerian* way by considering the velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ at each point $\mathbf{x}$, or we may use a *Lagrangian* description in which we follow the mass particles as time goes by, $\mathbf{x} = \mathbf{x}(t, \mathbf{a})$, $\mathbf{a} = \mathbf{x}(0, \mathbf{a})$.

A *material region* $R(t)$ is a section of the medium which at any time contains the same mass particles. Mathematically, $R(t)$ is defined as $R(t) = \{\mathbf{x}(t, \mathbf{a})\,;\ \mathbf{a} \in R(0)\}$, and a material region typically changes its shape and position as time passes, coinciding with the control volume at one instant of time, say at $t = 0$.

In this section we shall briefly review how the most important conservation laws of fluid mechanics may be derived by applying a simple result from vector analysis, namely *Reynolds' Transport Theorem*.

4.1 Reynolds Transport Theorem

From vector calculus we may know famous results such as the *Divergence Theorem* (also called Gauss' Theorem), and *Stokes' Theorem*. Reynolds Transport Theorem is another result in the same family. We introduce the theorem by first considering one-dimensional integrals.

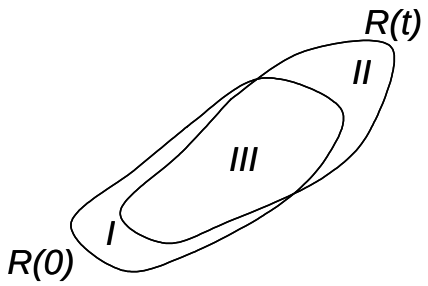

Figure 16: Definition of the regions I, II and III.

If we need to take the derivative w.r.t. to t of $H(t)$ defined by

$$H(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx, \tag{137}$$

it is possible first to write

$$H(t) = F(b(t),t) - F(a(t),t), \tag{138}$$

where F is the anti-derivative of f with respect to the first argument, and then apply the Chain Rule,

$$\frac{dH}{dt}(0) = \frac{d}{dt}\int_{a(0)}^{b(0)} f(x,t)\,dx\bigg|_{t=0} + f(b(0),0)\frac{\partial b}{\partial t}(0) - f(a(0),0)\frac{\partial a}{\partial t}(0). \tag{139}$$

This result is useful to know, and we see that in addition to the expected first term, we have extra contributions because the integration interval changes with time.

Reynolds transport theorem is this identity when integrating over a moving region in space. We shall assume that the region we are looking at, $R(t)$, is enclosed by a moving boundary, $\partial R(t)$. Furthermore, we assume that the points on the boundary are marked so that we can trace them as the time passes. In particular, all points on the boundary will, at any time, have a velocity $\mathbf{v}(\mathbf{x}(t))$, where $\mathbf{x}(t) \in \partial R(t)$.

We may then formulate Reynolds transport theorem for the integral of a function $\varphi(\mathbf{x},t)$ over the moving region $R(t)$ as follows:

$$\left(\frac{d}{dt}\int_{R(t)} \varphi(\mathbf{x},t)dV\right)_{t=0} = \left(\frac{d}{dt}\int_{R(0)} \varphi(\mathbf{x},t)dV\right)_{t=0} + \int_{\partial R(0)} \varphi(\mathbf{x},0)\mathbf{v}\cdot\mathbf{n}d\sigma. \tag{140}$$

The theorem requires that $\mathbf{v}$ and φ are sufficiently nice functions, and that $R(t)$ is a nice region, but we will not go into that here. The proof follows directly from the definition of the derivative. We assume $R(0)$ and $R(t)$ are as outlined in Fig. 16. Let $\Phi_i(t_0)$ denote the integral of φ over region "i" at time t, e.g.

$$\Phi_{III}(t) = \int\limits_{\mathbf{III}} \varphi(\mathbf{x},t)dV. \tag{141}$$

In general, the regions **I** in **II** are defined by the parts of $\partial R(0)$ where the velocity field points in and out of $R\,(0)$, respectively. From the definition of the derivative and Fig. 16 we have

$$\begin{aligned}\left(\frac{d}{dt}\int_{R(t)}\varphi(\mathbf{x},t)dV\right)_{t=0} &= \lim_{t\to 0}\frac{\Phi_{\mathbf{III}}(t)+\Phi_{\mathbf{II}}(t)-(\Phi_{\mathbf{I}}(0)+\Phi_{\mathbf{III}}(0))}{t}\\ &= \lim_{t\to 0}\frac{\Phi_{\mathbf{I}\cup\mathbf{III}}(t)-\Phi_{\mathbf{I}\cup\mathbf{III}}(0)}{t}+\lim_{t\to 0}\frac{\Phi_{\mathbf{II}}(t)}{t}-\lim_{t\to 0}\frac{\Phi_{\mathbf{I}}(t)}{t}. \qquad (142)\end{aligned}$$

The first limit value is just

$$\left(\frac{d}{dt}\int_{R(0)}\varphi(\mathbf{x},t)dV\right)_{t=0}. \qquad (143)$$

For small t the parts **I** and **II** become thin shells such that we may use the volume elements $dV=\mathbf{v}\cdot\mathbf{n}td\sigma$ for region **II,** and $dV=-\mathbf{v}\cdot\mathbf{n}td\sigma$ for region **I**. In the limit $t\to 0$ we obtain

$$\lim_{t\to 0}\frac{\Phi_{\mathbf{II}}(t)-\Phi_{\mathbf{I}}(t)}{t}=\int\limits_{\partial R(0)}\varphi(\mathbf{x},t)\mathbf{v}\cdot\mathbf{n}d\sigma\Bigg|_{t=0}. \qquad (144)$$

In many textbooks, one finds that the theorem is stated assuming

$$\frac{d}{dt}\int\limits_{R(0)}\varphi(\mathbf{x},t)dV=\int\limits_{R(0)}\frac{\partial\varphi}{\partial t}(\mathbf{x},t)dV. \qquad (145)$$

This is quite unfortunate for our applications and a direct error if φ has discontinuities inside $R\,(0)$ (recall the discussion in Sec. 2.5.4).

4.2 Mass Conservation

Mass conservation is a key principle in continuum mechanics. Here $\varphi=\rho$, i.e. the *mass* density of the medium. Without sources and sinks, the mass within a material region $R(t)$ will be constant, since this is precisely the definition of a material region. Thus,

$$\frac{d}{dt}\int\limits_{R(t)}\rho(\mathbf{x},t)dV=0, \qquad (146)$$

and, consequently, we get by applying the transport theorem

$$\frac{d}{dt}\int\limits_{R(0)}\rho(\mathbf{x},t)dV\Bigg|_{t=0}+\int\limits_{\partial R(0)}\rho(\mathbf{x},0)\mathbf{v}(\mathbf{x},0)\cdot\mathbf{n}(\mathbf{x},0)d\sigma=0. \qquad (147)$$

Since there is nothing particular with time $t=0$, we can for any time and an arbitrary fixed control volume R write

$$\frac{d}{dt}\int\limits_{R}\rho dV+\int\limits_{\partial R}\rho\mathbf{v}\cdot\mathbf{n}d\sigma=0. \qquad (148)$$

This is the mass conservation law in *integral form* when we do not have sources or sinks within R. Any sources/sinks will enter the expression as

$$\frac{d}{dt}\int\limits_R \rho dV + \int\limits_{\partial R} \rho \mathbf{v}\cdot\mathbf{n} d\sigma = \int\limits_R q dV. \tag{149}$$

and below this is discarded. Of course, we could see this immediately from the theory in Sec. 2.4, since mass is a material variable and the flux is $\rho\mathbf{v}$.

As discussed in Sec. 2.4 we can, when ρ and v are sufficiently smooth, differentiate under the integral sign and apply the Divergence Theorem:

$$\frac{d}{dt}\int\limits_R \rho dV = \int\limits_R \frac{\partial \rho}{\partial t} dV, \tag{150}$$

$$\int\limits_{\partial R} \rho \mathbf{v}\cdot\mathbf{n} d\sigma = \int\limits_R \nabla\cdot(\rho\mathbf{v}) dV \tag{151}$$

so that

$$\int\limits_R \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right) dV = 0. \tag{152}$$

Holding for all R, this then leads to the differential formulation

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{153}$$

The flow is called *stationary* if ρ and $\mathbf{v}$ are independent of time. Then $\int_R \rho dV$ will be constant and the mass conservation reduces to

$$\int\limits_{\partial R} \rho\mathbf{v}\cdot\mathbf{n} d\sigma = 0. \tag{154}$$

Equation 154 can be directly used for calculations, as illustrated for the pipeline in Fig. 17. By letting R be as given in the figure, the conservation law when the flow is stationary is

$$\int\limits_{A_1} \rho\mathbf{v}\cdot\mathbf{n} d\sigma + \int\limits_{A_2} \rho\mathbf{v}\cdot\mathbf{n} d\sigma + \int\limits_{A_3} \rho\mathbf{v}\cdot\mathbf{n} d\sigma = 0. \tag{155}$$

By assuming ρ is constant and defining the velocity over the cross section of the tubes,

$$v_i = \frac{1}{|A_i|}\int\limits_{A_i} \mathbf{v}\cdot\mathbf{n} d\sigma, \tag{156}$$

we obtain

$$|A_1|\,\rho_1 v_1 + |A_2|\,\rho_2 v_2 + |A_3|\,\rho_3 v_3 = 0. \tag{157}$$

($|A_j|$ is the cross sectional area). As one understands, it is easy to generalize this to arbitrary networks and otherwise include sources and sinks.

We immediately deduce the following special cases of the differential formulation:

- Stationary flow (no time variation): $\nabla\cdot(\rho\mathbf{v}) = 0$
- Constant density (*incompressible flow*): $\nabla\cdot\mathbf{v} = 0$

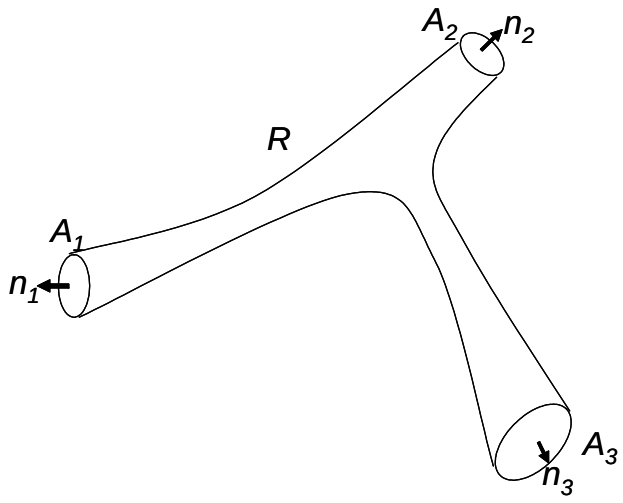

Figure 17: Stationary flow in a network of pipes will obey the mass conservation.

4.3 Momentum Conservation

In mechanics, *momentum* (or *linear momentum*) is the product of a body's mass times its velocity. Besides mass conservation, the momentum conservation law is the most important.

For a continuum, the *momentum density* is defined as momentum per volume unit, $\mathbf{p} = \rho\mathbf{v}$, where ρ is the mass density and $\mathbf{v}$ the velocity. The density is thus a three-dimensional vector in space. Newton's Second Law, stating that force is equal to mass times acceleration, is a statement about a fixed collection of mass particles. Since a material region $R(t)$ always contains the same mass particles, Newton's law applied to a material region $R(t)$ is just

$$\frac{d}{dt}\int\limits_{R(t)} \rho\mathbf{v}dV = \Sigma\mathbf{F}(t), \tag{158}$$

where $\Sigma\mathbf{F}(t)$ is the sum of all forced acting on the material region. By applying Reynolds transport theorem at $t = 0$ we obtain

$$\left(\frac{d}{dt}\int\limits_{R(0)} \rho\mathbf{v}dV + \int\limits_{\partial R(0)} (\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n})\,d\sigma\right)_{t=0} = \Sigma\mathbf{F}(0). \tag{159}$$

For a fixed control volume R, the conservation of momentum may therefore be expressed as

$$\frac{d}{dt}\int\limits_{R} \rho\mathbf{v}dV + \int\limits_{\partial R} (\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n})\,d\sigma = \Sigma\mathbf{F}. \tag{160}$$

The equation can be used to express the conservation of momentum in any direction: If $\mathbf{a}$

is a fixed unit vector in space, the scaler product with Eq. 160 gives

$$\frac{d}{dt}\int_R \rho v_a dV + \int_{\partial R} (\rho v_a)\,(\mathbf{v}\cdot\mathbf{n})\,d\sigma = \Sigma F_a, \tag{161}$$

where $v_a = \mathbf{a}\cdot\mathbf{v}$ and $F_a = \mathbf{a}\cdot\mathbf{F}$. For a Cartesian coordinate system in space, the three standard unit vectors give us three equations corresponding to the axes.

To get further, it is necessary to say something about forces acting on the mass in R. It is common to distinguish between *mass forces* (also called *body forces*) and *surface forces*. In general, it is possible to write

$$\mathbf{F}_B = \int_R \mathbf{f}_B(\mathbf{x},t)dV, \tag{162}$$

for mass forces and

$$\mathbf{F}_S = \int_{\partial R} \mathbf{f}_S(\sigma,t)d\sigma, \tag{163}$$

for surface forces.

The most common mass force is gravity,

$$\mathbf{F}_g = \int_R \rho\mathbf{g}dV. \tag{164}$$

In the geophysical fluid flow (such as oceans and atmosphere) the *Coriolis force* and the *centripetal acceleration* are important. These are forces that arise because our control volume is fixed on earth's surface and is thus rotating with the earth. The Coriolis force is given by

$$\mathbf{F}_c = \int_R \rho(-2\boldsymbol{\Omega}\times\mathbf{v})dV, \tag{165}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the earth ($2\pi/24\text{hours} = 7.3\times10^{-5}\text{s}^{-1}$). The centripetal acceleration

$$\mathbf{F}_s = \int_R \rho(\,-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}))dV, \tag{166}$$

where $\mathbf{r}$ is the position vector from the centre of the earth. Electromagnetic forces are other important examples of body forces.

The forces acting on the surface of R may be expressed in terms of the so-called *stress tensor* of the medium. Stress is the force per unit area. The force can act along a surface (*shear stress*) or orthogonal on the surface (*normal stress*). We refer to courses in mechanics for further discussion of the stress tensor. If we equip space with a Cartesian coordinate system, we may represent the stress tensor by a symmetric 3×3 matrix,

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}. \tag{167}$$

For a small surface element $d\sigma$ with normal vector $\mathbf{n}$ the force acting on $d\sigma$ is given by

$$d\mathbf{F} = \mathbf{T} \cdot \mathbf{n} d\sigma, \tag{168}$$

and the momentum conservation law in integral form may be written

$$\frac{d}{dt} \int_R \rho \mathbf{v} dV + \int_{\partial R} (\rho \mathbf{v}) \mathbf{v} \cdot \mathbf{n} d\sigma = \int_R \rho \mathbf{f}_B dV + \int_{\partial R} \mathbf{T} \cdot \mathbf{n} d\sigma. \tag{169}$$

If we are working with liquids, the stress tensor has contributions from pressure and viscosity forces. The pressure acts orthogonal on a small area element within the fluid and has the same value at a point no matter how the element is oriented. In addition, all common liquids are more or less viscous. Viscosity can be seen as a type of internal friction which provides resistance against deformations. For so-called *Newtonian fluids,* the shear stress in the x-direction for a flow with velocity $u(y)$ parallel to the x-axis is given by

$$\tau = \mu \frac{\partial u}{\partial y}, \tag{170}$$

where μ is called *dynamic viscosity.* It may be shown (See, *e.g.* [10]) from the mathematical properties of the stress tensor that the simplest expression consistent with Eq. 170 and giving the static pressure p when the fluid is at rest has to be of the form

$$t_{ij} = \left(-p - \frac{2}{3} \mu \nabla \cdot \mathbf{v} \right) \delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \ i, j = 1, 2, 3, \tag{171}$$

(indices refer to the standard Cartesian coordinate system). This expression leads to the quite famous equation for the momentum balance in a Newtonian fluid,

$$\frac{\partial}{\partial t} \rho \mathbf{v} + \nabla \cdot ((\rho \mathbf{v}) \mathbf{v}) = \rho \mathbf{f}_B - \nabla p + \mu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right), \tag{172}$$

called *Navier-Stokes Equation(s).* If the liquid is incompressible and the density is constant, the equation simplifies to

$$\frac{\partial}{\partial t} \mathbf{v} + \nabla \cdot ((\mathbf{v}) \mathbf{v}) = \mathbf{f}_B - \frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \mathbf{v}, \tag{173}$$

since $\nabla \cdot \mathbf{v} = 0$.

4.4 Energy Conservation

The first law of thermodynamics says that for a system in thermodynamic equilibrium, the added heat will be used to perform work and change the system's the internal energy,

$$dQ = dW + dE. \tag{174}$$

The energy may be expressed as specific energy e (energy per unit mass) so that

$$E(t) = \int_{R(t)} e\rho dV. \tag{175}$$

Contrary to Q and W, the specific energy e is a material variable. Specific energy may, *e.g.* consist of *kinetic* and *inner* energy per mass unit,

$$e = \mathbf{v} \cdot \mathbf{v}/2 + u. \tag{176}$$

Work performed by the system may be of different kinds. If we consider the work per unit time (power), we have

(i) *Work against the mass forces*:

$$\frac{dW_B}{dt} = -\int_R \mathbf{f}_B \cdot \mathbf{v} dV \tag{177}$$

(ii) *Work against surface forces*:

$$\frac{dW_S}{dt} = -\int_{\partial R} (\mathbf{T} \cdot \mathbf{n}) \cdot \mathbf{v} d\sigma \tag{178}$$

(iii) *Other work performed by the system (e.g. driving a turbine),*

$$\frac{dW_t}{dt}. \tag{179}$$

From the first law and Reynolds transport theorem, the general energy conservation law becomes

$$\frac{d}{dt}\int_R e\rho dV + \int_{\partial R} e\rho \mathbf{v} \cdot \mathbf{n} d\sigma = \frac{dQ}{dt} - \frac{dW_t}{dt} + \int_R \mathbf{f}_B \cdot \mathbf{v} dV + \int_{\partial R} (\mathbf{T} \cdot \mathbf{n}) \cdot \mathbf{v} d\sigma. \tag{180}$$

In the same way as for mass conservation, we can also derive the differential formulation using the Divergence Theorem, provided that the smoothness conditions are fulfilled. Further information may be found in textbooks about continuum mechanics.

4.5 Comments and Examples

There are several other conservation laws than those presented here. In particular, the conservation law for *vorticity* ($\nabla \times \mathbf{v}$) is important for many applications in fluid mechanics.

Traditional mechanics and mathematics teaching is oriented towards differential equations, i.e. differential formulations. This is natural since there is a huge theory about the existence of solutions, and techniques such as separation of variables, integral transforms, Green functions, and perturbation methods for finding solutions.

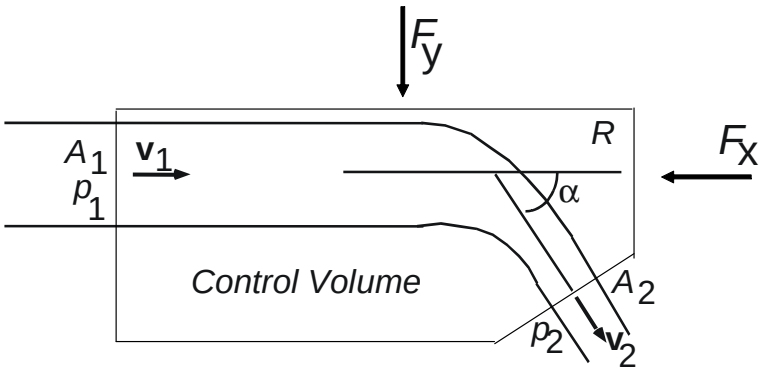

Figure 18: How much force should we apply to keep a bent tube in position?

Nevertheless, modern textbooks of practical mechanics to a greater extent base their arguments on integral formulations. The integral formulations are independent of the choice of coordinate system and embodies the fundamental physical laws (which, after all, manage the real world) more directly than differential formulations. Integral formulations may be used for practical tasks and a control volume need not be just a small box!

Conservation laws also apply to situations where differential equations have shortcomings, such as for discontinuous variables. This is especially important for treating shock solutions.

While numerical models traditionally have been made from differential equations by replacing the derivatives with the finite difference approximations, one can also use the integral formulation directly by dividing the computational region into a pile of boxes. The equations for each box are then established based on the conservation laws. This guarantees that the numerical solutions are compatible with the conservation laws. Finite Element formulations and so-called weak solutions of differential equations are also related to the conservation laws in integral form.

Below we will look at three examples of how one can operate with conservation laws. The first example should be familiar to anyone with some background in fluid mechanics.

The second example deals with the phenomenon of shock, and is typical for that type of problems. The conservation laws provide conditions that help us to determine the properties of the shock. In aerodynamics shocks are associated with supersonic speeds, while the hydrodynamic shock in the example occurs at the very mundane speeds. This is also the case for the third, somewhat more challenging, example.

4.5.1 Forces on a Pipe Bend

We consider a tube bend with stationary horizontal flow, see Fig. 18. We know the pressure, the cross-sectional area, density and the speed at both the inlet and outlet. The problem is to find the forces F_x and F_y that we must apply in order to keep the bend in position. The velocities are vectors with directions indicated by arrows, and we assume that the velocity

magnitude and the pressure are constant over the cross sections ($A_1$and A_2). Since the flow is stationary, the mass conservation requires

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2. \tag{181}$$

For the momentum balance we must first get an overview of the forces on R, which, in addition to the force needed to keep the bend in position, are composed of pressure forces:

$$\begin{aligned}(\Sigma F)_x &= p_1 A_1 - F_x - p_2 A_2 \cos\alpha, \\ (\Sigma F)_y &= -F_y + p_2 A_2 \ \sin\alpha.\end{aligned} \tag{182}$$

Since conditions are stationary,

$$\frac{d}{dt}\int\limits_R \rho \mathbf{v} dV = 0, \tag{183}$$

whereas

$$\begin{aligned}\int\limits_{\partial R} \rho v_x \mathbf{v}\cdot\mathbf{n} d\sigma &= \rho_1 v_1(-v_1 A_1) + \rho_2 v_2 \cos\alpha (v_2 A_2), \\ \int\limits_{\partial R} \rho v_y \mathbf{v}\cdot\mathbf{n} d\sigma &= \rho_1 \cdot 0 \cdot (-v_1 A_1) + \rho_2(-v_2 \sin\alpha)(v_2 A_2).\end{aligned} \tag{184}$$

If this is inserted into the conservation law, we obtain

$$\begin{aligned}F_x &= p_1 A_1 - p_2 A_2 \cos\alpha + \rho_1 v_1^2 A_1 - \rho_2 v_2^2 A_2 \cos\alpha, \\ F_y &= p_2 A_2 \sin\alpha + \rho_2 v_2^2 A_2 \sin\alpha,\end{aligned} \tag{185}$$

or, since $\rho_1 v_1 A_1 = \rho_2 v_2 A_2 = M$,

$$\begin{aligned}F_x &= M(v_1 - v_2 \cos a) + p_1 A_1 - p_2 A_2 \cos\alpha, \\ F_y &= M v_2 \sin\alpha + p_2 A_2 \sin\alpha.\end{aligned} \tag{186}$$

4.5.2 Flood Waves in Rivers

In this example we shall look at the simplest theory of flood waves and water jumps in rivers. Since the water level behind the water jump is higher than the level in front of the jump, people and livestock along the river can be swept away by the water, or suddenly find themselves in much deeper water than they appreciate. The reason for the jump could be torrential rain, sudden emission of water from a power station or a dam break.

To model what happens, we shall consider water flowing down a slope with a relatively small tilt angle α. The width of the flow across the inclined plane is B.

We assume that the velocity of the water is directed down the slope and has value v, is a function of position x and time t, and independent of z and y. This assumes that the flow is turbulent and that the water depth is not too large. In that case, the velocity is approximately constant over depth, but close to the bottom we will have a *boundary layer*

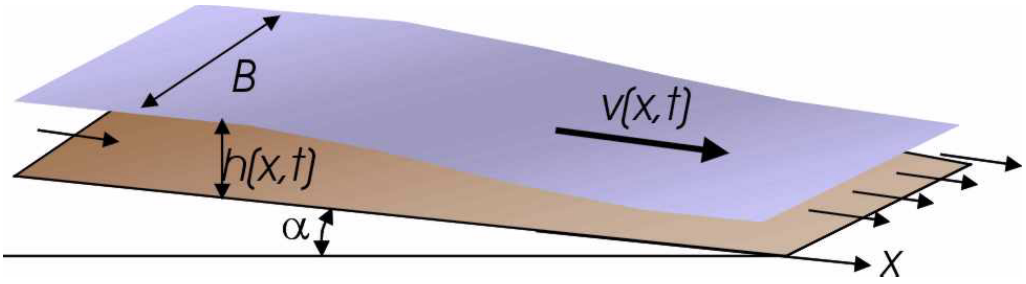

Figure 19: An idealized picture of a river.

where this assumption is not so good. Furthermore, we assume that the surface is defined by $z = h(x,t)$. To model the flow, we use mass and momentum conservation. The situation is illustrated in Fig. 19.

Let us first see what mass conservation provides. The conservation law has the general form

$$\frac{d}{dt}\int_R \rho dV + \int_{\partial R} \mathbf{j}\cdot\mathbf{n}d\sigma = 0. \tag{187}$$

We assume that the density ρ is constant and use $dV = Bh(x,t)dx$. The amount j that flows past a point x per unit time has the form

$$j(x,t) = \rho v(x,t)\cdot\left(Bh\left(x,t\right)\right). \tag{188}$$

The flux vector (amount per surface and unit time) is thus as expected

$$\mathbf{j}\left(x,t\right) = \rho v\left(x,t\right)\hat{\imath}. \tag{189}$$

This may be inserted into Eq. 187 for a section of the river between $x = a$ and $x = b$, while using $\mathbf{n} = -\hat{\imath}$ at $x = a$, and $\hat{\imath}$ in $x = b$:

$$\frac{d}{dt}\int_a^b \rho Bh(x,t)dx + \left[\left(\rho v\right)\left(Bh\right)\right]_a^b = \tag{190}$$

$$\frac{d}{dt}\int_a^b \rho Bh(x,t)dx + \left[\rho v(b,t)\cdot Bh\left(b,t\right) - \rho v(a,t)\cdot Bh\left(a,t\right)\right] = 0. \tag{191}$$

Note that the flux orthogonal to the bottom and the surface is zero, and also in the y-direction, since we assume no flow in that direction. By letting $a \to b$, dividing by $b-a$ in the usual way, and move the derivative inside the integral sign, we obtain the equation in differential form:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(vh\right) = 0. \tag{192}$$

Since this is an equation with two unknown functions, v and h, we can not solve the equation immediately.

The momentum balance will here give us something only for the x-direction, $p_x = \rho v$, and the conservation law takes the following form:

$$\frac{d}{dt}\int_a^b \left(\rho v\right)Bhdx + \left[\left(\rho v\right)v\cdot\left(Bh\right)\right]_a^b = \sum F_x. \tag{193}$$

It remains to specify the forces. Gravity acts directly on the water and, along the x-axis, this amounts to the force component proportional to $\sin\alpha$,

$$F_g = \int_a^b \rho g \sin\alpha B h\left(x,t\right) dx. \tag{194}$$

We then have the pressure forces. The pressure at the surface and the bottom does not contribute significantly to the x-component. However, we have a contribution from the end surfaces. Here we shall assume hydrostatic pressure, $p = \rho g\left(h-z\right)$, and sets $\cos\alpha \approx 1$ so that

$$dP = \rho g\left(h-z\right)\left(Bdz\right). \tag{195}$$

The total pressure force on the surface at $x = a$ is thus

$$P\left(a,t\right) = \int_0^h \rho g\left(h-z\right) Bdz = \rho g B\frac{h^2\left(a,t\right)}{2}, \tag{196}$$

and similarly at $x = b$, where the pressure force acts in the negative x-direction,

$$P\left(b,t\right) = -\rho g B\frac{h^2\left(b,t\right)}{2}. \tag{197}$$

The final force contribution is the friction force against the bottom. It turns out, partly based on dimensional analysis, that one can assume the friction force per area unit (*shear stress* in the x-direction) can be written

$$\tau = -\rho C_f \mathbf{v}\left|\mathbf{v}\right|. \tag{198}$$

The constant C_f is called the *Chézy-factor* and is empirically determined and depending of the roughness of the bottom. The total friction force is therefore found by integrating τ over the bottom surface:

$$F_f = -\int_a^b \rho C_f v^2\left(Bdx\right). \tag{199}$$

If we put all this together and divide by ρB, we get

$$\frac{d}{dt}\int_a^b vhdx + \left[\left(v\right)vh + \frac{g}{2}h^2\right]_a^b = \int_a^b \left(g\sin\alpha h - C_f v^2\right) dx. \tag{200}$$

It may be a bit tedious to establish the conservation law, but the principle is simple. The differential formulation follows in the same way as above:

$$\frac{\partial\left(hv\right)}{\partial t} + \frac{\partial}{\partial x}\left(v^2 h + \frac{g}{2}h^2\right) = g\sin\alpha h - C_f v^2. \tag{201}$$

The equations 192 and 201 are often called the *shallow water equations,* or *Saint-Venant Equations,* and constitute what is called a hyperbolic system. There is a theory for hyperbolic systems of two equations that we shall not go into here, but in general the equations

can not be solved analytically. However, it is easy to see that the equations have the solution

$$\begin{aligned} h(x,t) &= h_0, \\ v(x,t) &= v_0, \end{aligned} \tag{202}$$

where

$$g \sin \alpha h_0 - C_f v_0^2 = 0. \tag{203}$$

The last equation simply says that friction balances gravity. Linear stability analysis can tell whether the solution is stable, and this analysis, which is analogous to the one made for the instabilities in a traffic jam, may be found in the book by Whitham [18], p. 85–86.

In certain situations the flow is unstable, and more advanced analysis then leads to so-called *roll waves*. Roll waves may be observed on smooth sloping surfaces during heavy rain. Water flowing down a flat slope then has a tendency to create "waves" that are almost vertical in front and move slowly downward in relation to water velocity itself (see Fig. 22).

If we neglect the left side in Eq. 201 , gravity always balances the friction. This is called the *kinematic theory* of flood waves. We obtain a relation between h and v,

$$v = \sqrt{\frac{g \sin \alpha}{C_f}} h^{1/2}, \tag{204}$$

and Eq. 192 becomes

$$\frac{\partial h}{\partial t} + \sqrt{\frac{g \sin \alpha}{C_f}} \frac{\partial}{\partial x} h^{3/2} = 0. \tag{205}$$

The kinematic velocity is

$$c(h) = \frac{d}{dh} \sqrt{\frac{g \sin \alpha}{C_f}} h^{3/2} = \frac{3}{2} \sqrt{\frac{g \sin \alpha}{C_f}} h^{1/2} = \frac{3}{2} v(h). \tag{206}$$

We leave to the reader to show that if the water level in the upper part of a river increases, in other words, if $\partial h / \partial x < 0$, it may develop a shock that in this case moves down the river like a wall. The phenomenon can occur during torrential rain or in rivers with regulated water flow, such as in rivers downstream from power plants. Note that the speed of the shock will be about 50% greater than the speed of the water flow!

On the figures 20 and 21, downloaded from the WEB-page of Dr. Hubert Chanson, Univ. of Queensland, Australia (`http://www.uq.edu.au/~e2hchans/`), we see examples of similar phenomena. These waves are called *tidal bores*, and occur as shock-solutions for the equations 192 and 201 in a flat river when the tide enters the river from its mouth. Fig. 20 is the most famous example of a tidal bore.

Figure 21 shows a somewhat different case. The picture is from the river Dorgonge in France. The river Seine had in the past a similar wave that could reach in some cases all the way to Paris, but this does not appear to be that prominent anymore.

The last image (Fig. 22) is copied from A. C. Fowler's photo gallery and shows *roll waves* on a road.

Figure 20: The tidal bore on Qiantang River near Hangzhou, China, also known as Hang chow or *Hangzou Bore*. The wave can be up to 9 feet tall and enters 2 times daily. It is most pronounced around the spring and autumn equinox (© Dr. H. Chanson).

Figure 21: Surfers on the tidal bore entering the Dordogne (© Dr. H. Chanson).

Figure 22: *Roll waves* on an asphalt road in the rain (Image copied from `http://www.maths.ox.ac.uk/~fowler/pictures/gallery.html`).

4.5.3 Research Project: The Circular Water Jump

Everyone who has run tapped water vertically into the kitchen sink, has observed that a circular water jump often forms some distance from where the jet hits the surface. If this does not sound familiar, one should before reading further make a simple experiment as in Fig. 23 The geometry of the problem is indicated in Fig. 24.

We assume radial symmetry and a constant density ρ. Furthermore, we assume that the water velocity is directed radially outward and is independent of z. Thus, both speed and depth are only functions of r and t. As for the pressure, we assume that this is given by the hydrostatic pressure, $p(r,t,z) = \rho g(h(r,t) - z)$ since any constant atmospheric pressure drops out. Frictional force per area unit at the bottom has the form $\mathbf{t}_C = -C_f \rho \left|\mathbf{v}\right| \mathbf{v}$ (where again C_f is the Chezy friction factor).

Let us set up the general conservation laws of mass and momentum for a control volume limited by r_1 and r_2, $r_1 < r_2$, where r_1 is greater than the radius of the center jet. From the mass conservation law,

$$\frac{d}{dt}\int\limits_R \rho dV + \int\limits_{\partial R} \rho \left(\mathbf{v}\cdot\mathbf{n}\right) d\sigma = 0, \tag{207}$$

we immediately get, using polar coordinates and dividing by $2\pi\rho$,

$$\frac{d}{dt}\int\limits_{r=r_1}^{r_2} h(r,t) r dr + r_2 h_2 v_2 - r_1 h_1 v_1 = 0. \tag{208}$$

Figure 23: When a vertical water stream hits a horizontal surface, a circular jump in the flow forms some distance away from where the stream hits.

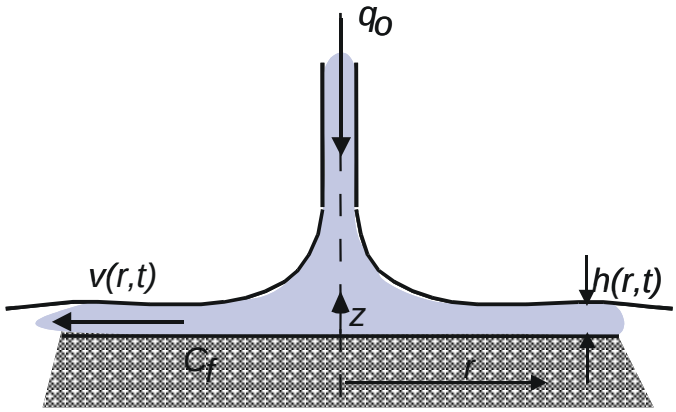

Figure 24: Vertical water jet hitting a horizontal plane.

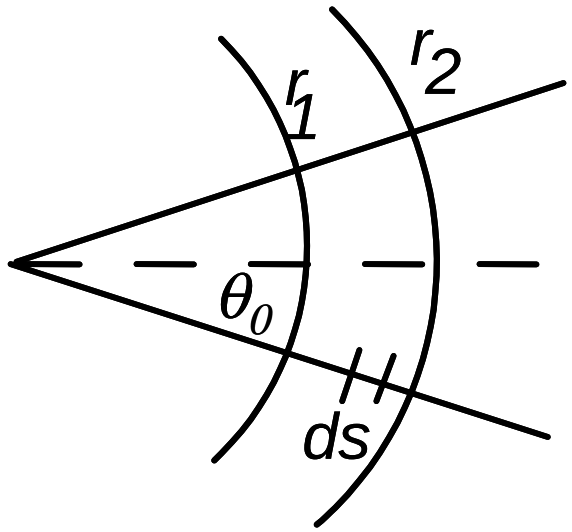

Figure 25: Sector-shaped control volume for the momentum balance.

For the momentum balance it is necessary to choose a pie-shaped section as control volume and compute, *e.g.* the momentum balance in the x-direction, see Fig. 25.

Since the pressure on the bottom surface works vertically, it is sufficient to look at the pressure forces on the side walls. The pressure forces acting on a strip of width ds of the side wall is $dP = \rho g h^2(r,t)ds/2$, and by integrating around all side walls we obtain

$$P_x = \int\limits_{\text{Sides}} -pn_x d\sigma = \rho g \sin\theta_0 \left(h^2(r_1,t)r_1 - h^2(r_2,t) + \int\limits_{r=r_1}^{r_2} h^2(r,t)dr \right). \tag{209}$$

For the bottom friction we have, as for the flood waves,

$$C_x = -\int\limits_{r=r_1}^{r_2}\int\limits_{\theta=-\theta_0}^{\theta_0} C_f \rho v \cdot (v\cos\theta) r dr d\theta = -2C_f \rho \sin\theta_0 \int\limits_{r=r_1}^{r_2} v^2(r,t) r dr. \tag{210}$$

The rest of the expressions is left to the reader, and we finally end with

$$\frac{d}{dt}\int\limits_{r_1}^{r_2} vhr dr + \left[v^2 rh + \frac{r}{2}h^2 g \right]_{r_1}^{r_2} = \int\limits_{r_1}^{r_2} \left(-C_f v^2 r + \frac{h^2}{2} g \right) dr. \tag{211}$$

Differential formulations of mass and momentum follow in the usual way:

$$\begin{aligned} \frac{\partial(rh)}{\partial t} + \frac{\partial(rhv)}{\partial r} &= 0, \\ \frac{\partial(rhv)}{\partial t} + \frac{\partial}{\partial r}(rhv^2 + rh^2 g/2) &= -C_f v^2 r + \frac{h^2}{2} g. \end{aligned} \tag{212}$$

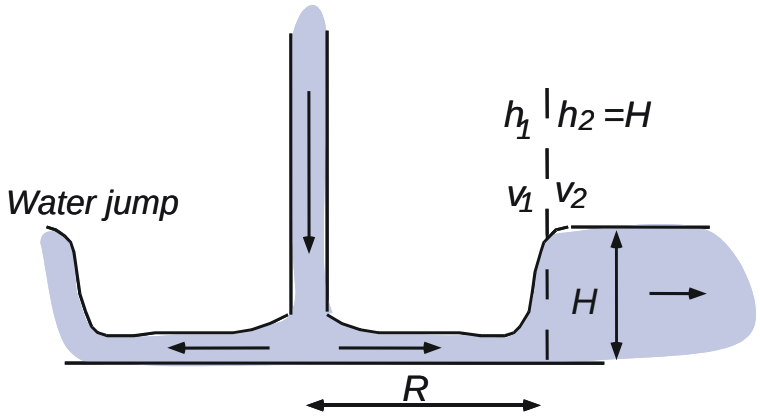

Figure 26: Formation of a circular water jump.

A suitable outflow will produce a stationary annular jump some distance from where the jet strikes, as indicated in Fig. 26. The water jump is actually a shock called a *hydraulic jump*. By assuming steady flow and letting $r_1 \to r_2 = R$ in the conservation laws, all the integrals of the conservation laws disappear, and we end up with the following classical shock conditions derived by J.-B. Belanger in 1838 [17]):

$$v_1 h_1 = v_2 h_2. \tag{213}$$

$$v_1^2 h_1 + \frac{1}{2} h_1^2 g = v_2^2 h_2 + \frac{1}{2} h_2^2 g. \tag{214}$$

This is not sufficient to determine the position of the shock. We refer to the fluid mechanics text books for an discussion of the energy conservation in an ideal fluid under stationary conditions, leading to *Bernoulli's equation*, stating that the quantity

$$gz + \frac{v^2}{2} + \frac{p}{\rho} \tag{215}$$

is constant along streamlines. For a streamline on the surface (or at the bottom) this gives a third condition at the jump:

$$gh_1 + \frac{v_1^2}{2} = gh_2 + \frac{v_2^2}{2}. \tag{216}$$

In the theory of hyperbolic conservation laws, conditions such as the ones stated in the equations 213, 214 and 216 are called *Rankine-Hugoniot conditions.*

The equations 213, 214 and 216 have the obvious and not particularly exciting solution $h_1 = h_2,\ v_1 = v_2\ ,\ h_L = 0$. However, if one requires that $h_1 \neq h_2$, the three equations do not have a solution. In reality, some of the energy is transformed to turbulence and eventually to heat at the shock. This requires the energy condition to include a certain energy loss at the jump:

$$gh_1 + \frac{v_1^2}{2} = gh_2 + \frac{v_2^2}{2} + gh_L. \tag{217}$$

The extra quantity gh_L on the right side is called *head loss* and is basically unknown.

If we only consider 213 and 214, it is possible to derive the equation

$$\left(\frac{h_2}{h_1}\right)^2 + \frac{h_2}{h_1} = 2\frac{v_1^2}{gh_1}. \tag{218}$$

The dimensionless combination

$$Fr = \frac{v}{\sqrt{gh}}. \tag{219}$$

occurring on the right hand side is called *Froude's Number*, and is an important number in hydrodynamics. The quadratic equation for h_2/h_1 has the solution

$$\frac{h_2}{h_1} = \frac{1}{2}\left(-1 \pm \sqrt{1+8Fr_1}\right) \tag{220}$$

For h_1 to be less that h_2, it is necessary that $Fr_1 > 1$. The Froude number has an interesting physical interpretation. Water waves (longer than a few centimeters) have the propagation speed $c_p = \sqrt{gh}$ in shallow water. Thus, the Froude number is the ratio between v and c_p. A flow with a free surface where $v > c_p$ is called *supercritical flow*. If you are sitting in the flow, a disturbance in front of you cannot be warned by a surface wave before it happens to you. Thus, *Fr* is analogous to the *Mach number* in aerodynamics.

From the energy condition, we can find an expression for the relative energy loss in the shock expressed as the ratio h_L/h_1:

$$\begin{aligned}\frac{h_L}{h_1} &= \frac{h_1 - h_2 + v_1^2/2g - v_2^2/2g}{h_1}, \\ &= 1 - \frac{h_2}{h_1} + \frac{Fr^2}{2} - \frac{v_2^2}{2gh_1} = 1 - \frac{h_2}{h_1} + \frac{Fr^2}{2}\left(1 - \left(\frac{1}{h_2/h_1}\right)^2\right).\end{aligned} \tag{221}$$

where we finally may insert Eq. 220.

But what is the position of the water jump? A simplified analysis is found in [17], ignoring all other energy losses apart from at the shock. If the kinetic energy in the flow before the shock is much larger than potential energy, the velocity, according to Bernoulli's equation, will be approximately constant and equal to U_0, i.e. the speed of the jet as it hits the plate. We then obtain from the mass and momentum, balance

$$\begin{aligned} q_0 = 2\pi R h_1 U_0 = 2\pi R H v_2, \\ U_0^2 h_1 + h_1^2 g/2 = v_2^2 H + H^2 g/2, \end{aligned} \tag{222}$$

which after a simple transformation gives

$$R = \frac{(U_0^2 - gH/2)q_0}{\pi g H^2 U_0}. \tag{223}$$

It is not unreasonable that U_0 enters in addition to q_0, since both the added momentum and mass should be of importance for the position of the jump.

In September 1993, *Journal of Fluid Mechanics* presented a comprehensive analysis of the problem [3].

5 DIFFUSION AND CONVECTION

Whereas a material variable is a quantity passively transported along with the flowing medium, this is not always a reasonable assumption. On a small scale, the *molecular diffusion* mixes liquids and gases by the molecules tumbling around. This also applies if we add small particles of a liquid (mixture by *Brownian motion*). There are differences in the concentration that give rise to the mixing. If the medium is moving, we will also have a change because of transportation. Such transport is called *convection*.

Low viscosity fluids will often have speeds and length scales so that *turbulence* is developed. Turbulence is the chaotic movement where the flow develops vortices and thin layers that spin into each other and split up (smoke in the air visualizes this well). If we add a foreign substance to a medium, we observe that the mixing goes much faster when the flow is turbulent than it does if the mixture occurs by diffusion alone (this is why we use a spoon in the cup in order to mix milk and coffee!). Modeling by turbulent mixing is a classic problem. The simplest would then be trying to describe it as an enhanced diffusion, but there are also examples where such a description appears to be quite wrong. Turbulence in fluids can occur when the speed varies greatly from place to place (called the velocity shear in fluid mechanics). Otherwise, the turbulence may be caused by temperature and density differences (always observed when we boil water). In Chapter 6, we shall derive the conservation laws for turbulence from the conservation laws in Sec. 4

In addition to molecular diffusion and turbulence, diffusion-like spread has been shown to be very applicable in other contexts. Often, the offspring is a discrete phenomenon, modeled by stochastic random walk. Mathematically, one can then show that such discrete models in the limit with very many objects actually transform into diffusion models.

5.1 Conservation Laws with Diffusion

Diffusion is flux caused *concentration differences*. If the concentration is constant, there will be no net flux in any direction. To first order, the flux must be proportional to the change in concentration per unit length, in other words

$$\mathbf{j} = -\sigma \nabla \varphi, \tag{224}$$

where φ is the concentration and σ is called the *diffusion coefficient.* This expression is called *Fick's Law* for diffusion. Heat conduction obeys a similar law. If the specific heat is constant, the heat flux is $\mathbf{q} = -k\nabla T$, where T is the temperature and k is called the *heat conduction coefficient*. This expression is called *Fourier's Heat Conduction Law.*

Diffusion has a *smoothing* effect on the concentration gradients. If we start with a localized section R of a liquid with a different concentration of some substance, the largest concentration gradients occur along the edges of R. When we stir and develop turbulence, the batch of liquid is stretched and twisted so that an increasing proportion is located in areas with strong gradients. In this way, diffusion acts more strongly, and this is the mechanism behind what we call *forced mixing* or enhanced diffusion by turbulence.

Let us now consider a general situation where $c(\mathbf{x},t)$ is the concentration of a substance in a liquid, and the vector field $\mathbf{j}(\mathbf{x},t)$ denotes the corresponding flux. If the substance passively follows the flow, we have shown before that the flux is just $\mathbf{j}_c(\mathbf{x},t) = c\mathbf{v}$. Even when the liquid is at rest, the material may be spread by a diffusion flux $\mathbf{j}_d$. We want to show that the total flux will be the sum of the fluxes, $\mathbf{j}(\mathbf{x},t) = \mathbf{j}_d(\mathbf{x},t) + \mathbf{j}_c(\mathbf{x},t)$. Let $R(t)$ be a material region of the liquid much smaller than the scale of variations in $\mathbf{v}$. For time intervals of the order of the diameter of R divided by $|\mathbf{v}|$, an observer traveling with $R(t)$ set up

$$\frac{d}{dt}\int\limits_{R(t)} c\,(\mathbf{x},t)\,dV\Bigg|_{t=0} + \int\limits_{\partial R(0)} j_d\,(\mathbf{x},0)\cdot\mathbf{n}d\sigma = \int\limits_{R(0)} q\,(\mathbf{x},t)\,dV. \tag{225}$$

But Reynolds transport theorem applied to the first term gives

$$\frac{d}{dt}\int\limits_{R(t)} c\,(\mathbf{x},t)\,dV\Bigg|_{t=0} = \frac{d}{dt}\int\limits_{R(0)} c\,(\mathbf{x},t)\,dV + \int\limits_{\partial R(0)} c\,(x,0)\,\mathbf{v}\cdot\mathbf{n}d\sigma, \tag{226}$$

and altogether,

$$\frac{d}{dt}\int\limits_{R} cdV + \int\limits_{\partial R} (\mathbf{j}_d + c\mathbf{v})\cdot\mathbf{n}d\sigma = \int\limits_{R} qdV. \tag{227}$$

A general control region can be divided into arbitrary small parts where this formula holds. When all contributions are added, the area integrals over common borders cancel so that one ends up with the same formula for the whole region. This is thus the general diffusion conservation law, and the total flux is

$$\mathbf{j}(\mathbf{x},t) = \mathbf{j}_d(\mathbf{x},t) + \mathbf{j}_c(\mathbf{x},t). \tag{228}$$

If we apply the Divergence theorem as in Chapter 4, we find the differential form:

$$\frac{\partial c}{\partial t} + \nabla\cdot(\mathbf{j}_d + \mathbf{v}c) = q. \tag{229}$$

Inserting for $\mathbf{j}_d = -\sigma\nabla c$, we get a *Convection/diffusion Equation*:

$$\frac{\partial c}{\partial t} + \nabla\cdot(\mathbf{v}c) - \nabla\cdot(\sigma\nabla c) = q. \tag{230}$$

In general, both $\mathbf{v}$ and σ may depend on c so that the equation is nonlinear. In the next section we will consider a simple mathematical model where this equation enters in a central way.

5.2 One–Dimensional Chemical Reactor

A simple chemical reactor consists of a tube (often filled with crushed glass or glass spheres) where a fluid is flowing with a constant mean velocity v. A certain substance is added to

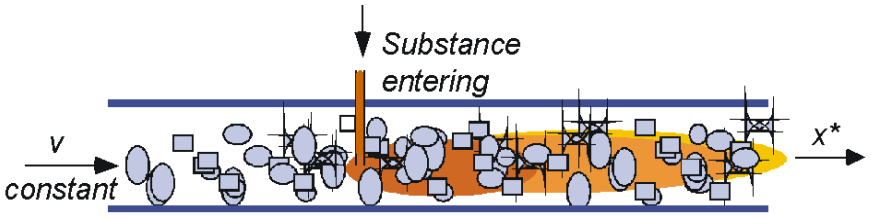

Figure 27: Sketch of a simple one-dimensional chemical reactor. The fluid flows through a tube filled with crushed glass or glass spheres.

the fluid from a nozzle. The is mixed with a constant diffusion coefficient σ. In practice, σ is an *effective* diffusion coefficient due to the turbulent mixing in the flow around the glass), as illustrated in Fig. 27. The concentration of the substance (outside the source region) is described by a one-dimensional version of Eq. 230:

$$\frac{\partial c}{\partial t^*} + v\frac{\partial c}{\partial x^*} - \sigma\frac{\partial^2 c}{\partial x^{*2}} = 0. \tag{231}$$

Scaling x^* by the typical tube length L and a corresponding time scale $T = L/v$ gives us

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} - \varepsilon\frac{\partial^2 c}{\partial x^2} = 0, \tag{232}$$

where $\varepsilon = \sigma/Lv$ is a dimensionless parameter, similar to the inverse Reynolds number. When ε is small, we have an equation with a small parameter in front of the highest derivative. This is the case when convection is dominating over diffusion. When $\varepsilon > 0$, the equation belongs to the *parabolic* class of PDEs, whereas it is *hyperbolic* when $\varepsilon = 0$. In this case, it is not just the order of the equation that changes, the type changes as well.

Let us consider the solution of the initial value problem:

$$\begin{aligned} \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} - \varepsilon\frac{\partial^2 c}{\partial x^2} = 0,\ t > 0,\ -\infty < x < \infty, \\ c(x,0) = f(x) \end{aligned} \tag{233}$$

For $\varepsilon = 0$ the equation is $c_t + c_x = 0$, and we have already seen in Sec. 2.5 that the solution is then simply

$$c(x,t) = f(x-t). \tag{234}$$

The initial density profile moves unchanged to the right with speed 1 (speed v in the original variables). This is also reasonable from a purely physical reasoning.

When $\varepsilon > 0$, it is convenient to choose a coordinate system following the flow:

$$\begin{aligned} x' &= x - t, \\ t' &= \varepsilon t. \end{aligned} \tag{235}$$

Then

$$\begin{aligned}
\frac{\partial c}{\partial t} &= \frac{\partial c}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial c}{\partial x'}\frac{\partial x'}{\partial t} = \varepsilon c_{t'} - c_{x'}, \\
\frac{\partial c}{\partial x} &= \frac{\partial c}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial c}{\partial x'}\frac{\partial x'}{\partial x} = c_{x'}, \\
\frac{\partial^2 c}{\partial x^2} &= c_{x'x'}.
\end{aligned} \tag{236}$$

Thus,

$$c_t + c_x - \varepsilon c_{xx} = \varepsilon c_{t'} - c_{x'} + c_{x'} - \varepsilon c_{x'x'} = 0, \tag{237}$$

or

$$c_{t'} = c_{x'x'}, \tag{238}$$

which is the classical parabolic equation. We leave to the reader to show that the function

$$c_f(x', t') = \frac{1}{\sqrt{4\pi t'}} \exp\left(-\frac{x'^2}{4t'}\right) \tag{239}$$

is a solution for $t' > 0$. This is the so-called *fundamental solution*. From the formula

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \tag{240}$$

we find that

$$\int_{-\infty}^{\infty} c_f(x', t') dx' = 1 \tag{241}$$

for all $t' > 0$. The fundamental solution is, in other words, a Gaussian distribution with variance increasing linearly with time. When $t' \to 0$, the solution approach a δ-function at the origin. Physically, this corresponds to a release of a unit quantity of the substance at the origin at time $t' = 0$.

If we return to original coordinates, we obtain

$$c_F(x, t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \exp\left(-\frac{(x-t)^2}{4\varepsilon t}\right). \tag{242}$$

The solution is sketched in Fig. 28. It is easy to show that the general solution of the initial value problem $c(x, 0) = f(x)$ can be expressed as a *convolution integral* with the fundamental solution:

$$c(x, t) = \int_{s=-\infty}^{\infty} f(s) c_F(x - s, t) ds. \tag{243}$$

A general feature of convolution says that the result is at least as nice as the nicest of the functions involved, and actually, the solution $c(x, t)$ will be infinitely many times differentiable for $t > 0$ no matter how f looks. Fast variations are smoothed out more and more as the time goes on, and

$$\lim_{t\to\infty} c(x, t) = 0. \tag{244}$$

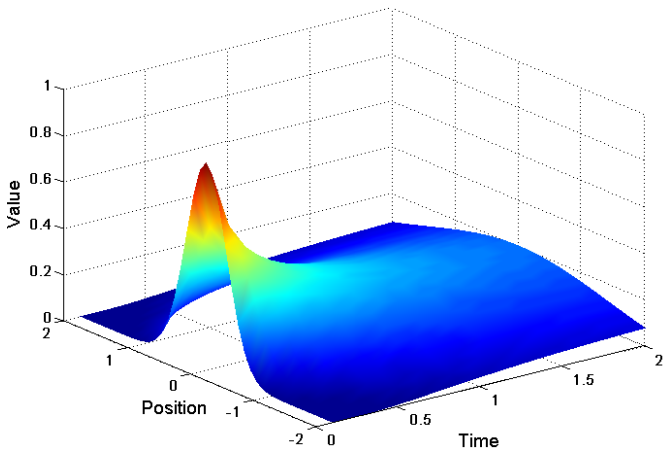

Figure 28: Time development of the fundamental solution shown for $t = 0.1$ up to 2, and $\varepsilon = 1$.

How fast smearing takes place depends on the size of ε. In this linear equation, the solution becomes smoother and smoother as time passes, but for more general non-linear convection/diffusion equations this does not need to be the case.

On its way towards the final stage, the solution of Eq. 243 passes what is called an *intermediate asymptotic* state. The concept was introduced in [1] and says (in the simplest case) that the solutions of differential equations asymptotically may approach simplified solutions as time passes, and before they reach their final stage.

Consider the diffusion equation

$$\frac{\partial c^*}{\partial t^*} = \kappa \frac{\partial^2 c^*}{\partial^2 x^*}, \; 0 < t^*, \; -\infty < x^* < \infty, \tag{245}$$

with initial condition $c^*(x^*, 0) = f(x^*)$, and where f is a function localized to the interval $[-L, L]$ such that

$$\int_{-L}^{L} f(x^*) dx^* = Q_0, \tag{246}$$

$$f(x^*) = 0 \text{ for } L \leq |x^*|. \tag{247}$$

For large times, $t = \mathcal{O}(T)$, the solution, due to diffusion, has a spatial extension $X = \mathcal{O}(\sqrt{\kappa T})$, where $X \gg L$ (However, mathematically the solution is non-vanishing on the whole interval $[-\infty, \infty]$ for all $t > 0$).

Let us scale the problem by T, X and Q_0/X:

$$\begin{aligned} t^* &= Tt, \\ x^* &= Xx = \sqrt{\kappa T} x, \\ c^* &= \frac{Q_0}{X} c. \end{aligned} \tag{248}$$

This gives

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial^2 x}, \tag{249}$$

while the initial condition is now

$$f(x) = c(x,0) = \frac{X}{Q_0} c^*(xX,0) = \frac{X}{Q_0} f^*(Xx). \tag{250}$$

We see that $f(x) = 0$ for $|x| > \varepsilon = L/X \ll 1$, and

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{Q_0}\int_{-\infty}^{\infty} f^*(Xx)Xdx = 1. \tag{251}$$

The scaled problem becomes

$$c_t = c_{xx},\ c(x,0) = f(x), \tag{252}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1,\ f(x) = 0 \text{ for } |x| > \varepsilon. \tag{253}$$

At $t = O(1)$, the initial condition looks like a δ-function regardless of how irregular f is, and the actual solution must therefore be quite similar to the fundamental solution (The argument can be made mathematically precise by studying the convolution integral in Eq. 243). Fundamental solutions therefore have greater applicability than one might think:

For large times, the details of the initial conditions are blurred and we can use solutions derived from the simpler conditions.

Linear diffusion equations in one or several dimensions are thoroughly covered in all textbooks on partial differential equations.

5.3 A Nuclear Power Plant Accident

In this case study we show how one can use conservation principles and basic properties of the fundamental solution of the one-dimensional convection/diffusion equation to analyze a hypothetical release of radioactive material. The analysis is typical of how one will try to get a first rough overview of a relatively difficult modeling problem. The situation is of course fictional, and was an exam task at NTNU in 1986, just after the Chernobyl disaster.

From a nuclear power plant there is an uncontrolled release of radioactive cooling water to a river past the power plant. The radioactivity is mainly due to a certain short-lived isotope (Numerical values below are selected in order to produce reasonably simple numeric answers).

The special thing here is that the radioactive material breaks down. If we have a solution with a concentration c of radioactive material, the concentration decays exponentially with time,

$$c(t) = c(0)\exp(-t/t_0). \tag{254}$$

This could also be described by the differential equation

$$\frac{dc}{dt} = -\frac{c}{t_0} \tag{255}$$

where the time $t_0 \ln 2$ is the *half-life* for the isotope.

We assume that the river stream flows at a mean velocity $U = 0.2\text{m/s}$. In reality, the water velocity varies with the river topography, and the waters are mixed and spread both by turbulence and because the water velocity is not constant over the cross section of the river. In practice, it is common to model this by a so-called *eddy-diffusivity* (*eddy* = whirl) along the direction of the river, defined by a diffusion coefficient κ_E 6.The diffusion coefficient will have the same dimension as the molecular diffusions coefficient, but will be much larger. Here, we let κ_E be equal to $1\text{m}^2/\text{s}$, which is not a completely unreasonable value. We consider the river to be one-dimensional and assume that emissions come from from a stationary point-source.

When formulating the conservation law, the density of radioactive material c enters as amount per length unit of the river, while the flux will have two contributions, one from the diffusion and one from convection:

$$J(x,t) = c(x,t)U - \kappa \frac{\partial c}{\partial x}(x,t). \tag{256}$$

The decreasing radioactivity can be modeled as a sink with intensity $\frac{c}{t_0}$. The spill is a δ-function source in $x = 0$. The change of radioactive material within an interval $[x_1, x_2]$ of the river is thus described by the conservation law

$$\frac{d}{dt}\int_{x_1}^{x_2} c(x,t)\,dx + J(x_2,t) - J(x_1,t) = \int_{x_1}^{x_2} \left(-\frac{c(x,t)}{t_0} + q(t)\delta(x) \right) dx. \tag{257}$$

The differential form of the the equation will be

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} - \kappa \frac{\partial^2 c}{\partial x^2} = -\frac{c}{t_0} + q(t)\delta(x), \tag{258}$$

which is a linear convection/diffusion equation with a source/sink term. Assume that the discharge has been going on with a constant amount q_0 per unit of time from time $t = 0$. The total amount of radioactive material in the river at any time can then be calculated from the conservation law by integrating from $x_1 = -\infty$ to $x_2 = \infty$:

$$\begin{aligned} \frac{dC}{dt} &= -\frac{C}{t_0} + q_0, \\ C(t) &= \int_{-\infty}^{\infty} c(x,t)\,dx. \end{aligned} \tag{259}$$

Note that the flux terms disappear since $\lim_{x\to\infty} J(x,t) = \lim_{x\to-\infty} J(x,t) = 0$. The solution for $C(t)$ under the assumption that $C(0) = 0$ follows immediately:

$$C(t) = q_0 t_0 \left(1 - \exp(-t/t_0)\right). \tag{260}$$

As $t \to \infty$, the total amount converges to $q_0 t_0$.

Consider now a situation where a constant discharge q_0 lasts for $t_1 = 30$ minutes. We are seeking an approximate solution for $c\,(x,t)$ at two different times: (i) immediately after the spill is over, and $(ii\ )$ after $t_2 = 10^6\text{s} \approx 11.5$ days.

After the 30 minutes are over, there is a total amount

$$C_0 = t_0 q_0 \left(1 - e^{-t_1/t_0}\right) \approx t_1 q_0 \tag{261}$$

in the water (note that $t_1 << t_0$). Convection (i.e. the motion of water masses with mean speed U) has led to a spreading of material over a length $L = Ut_1 = 360$m. How much diffusion has affected the solution is estimated by the length scale for diffusion,

$$\sigma_1 = \sqrt{2\kappa t_1} = 60\text{m}. \tag{262}$$

This is significantly less than L. Since there is no appreciable radioactive decay during this short period, the concentration, c_0, is nearly constant from $x = 0$ to $x = L$, and mass balance gives $c_0 L = q_0 t_1$. Thus,

$$c(x, t_1) \approx \begin{cases} c_0 = q_0/U, & 0 \le x \le L \\ 0, & \text{otherwise.} \end{cases} \tag{263}$$

Actually, the solution should be a little "rounded" on both ends, but we have a good approximation when neglecting both radioactive decay and turbulent diffusion.

After 10^6 s the dispersion and decay can no longer be neglected. The length scale for the diffusion is now

$$\sigma_2 = \sqrt{2\kappa t_2} \approx 1400\text{m}, \tag{264}$$

which is significantly larger than the original length L of the discharge. The total amount of radioactive material is given by

$$\int_{-\infty}^{\infty} c(x, t_2)\, dx = \left[q_0 t_1 (1 - e^{-t_1/t_0}\right] e^{-(t_2 - t_1)/t_0} \approx q_0 t_1 \exp(-t_2/t_0) \tag{265}$$

Relative to the amount just after the end of the discharge, the remaining amount is about $\exp(-t_2/t_0) \approx 0.8 \cdot 10^{-12}$ less.

Since $t_1 = 30$minutes is much less than $t_2 = 10^6$s, and $L/\sigma_2 \approx 0.25$, we can, with high accuracy, assume that all emissions of radioactive material occurred at time zero. It is then possible to exploit the fundamental solution to the convection/diffusion equation stated in Eq. 242 giving the solution of a unit discharge at $t = 0$ and $x = 0$. The full solution becomes, approximately,

$$c(x, t_2) \approx \left[q_0 t_1 \exp(-t_2/t_0)\right] \frac{1}{\sqrt{4\pi\kappa t_2}} e^{-(x - Ut_2)^2/(4\kappa t_2)}, \tag{266}$$

where the first part denotes the total remaining radioactive material and $Ut_2 = 200$km.

It is actually easy to write down the solution for an arbitrary time-variable discharge $q(t)$ from a point-source. Since we can assume that the emissions at different times do not affect each other, it is possible to consider emissions as a series of point-discharges, and then sum up the corresponding solutions. In the same way as above, the solution of a discharge over time duration $d\tau$ at a time $\tau < t$

$$dc(x, t) = \left[q(\tau) d\tau \exp(-(t - \tau)/t_0)\right] F(x, t - \tau). \tag{267}$$

The distribution of radioactive material at time t is then given by the integral

$$c(x,t) = \int_{-\infty}^{t} dc(x,t) = \int_{-\infty}^{t} \left[q(\tau)d\tau \exp(-(t-\tau)/t_0)\right] F(x,t-\tau)d\tau. \tag{268}$$

In practice, the radioactive emissions often include several different isotopes. Decomposition of one isotope could also lead to other radioactive isotopes. This will lead to connections between the conservation laws for the individual isotopes, but the link is limited to the source-terms. Moreover, the modeling of the river will naturally also be made considerably more advanced. In particular, turbulent mixing is described by models that relate the strength of the diffusion of the amount of turbulent kinetic energy. The modeling will first calculate the flow and turbulence level in the water using a hydrodynamic turbulence model. Then run a *transport model* calculating the distribution of the radioactive material on the basis of the hydrodynamic solution.

Today, the authorities require that such models are developed and tested *before* an accident occurs (the models can actually be tested by controlled release of radioactive isotopes).

5.4 Similarity Solutions

So far we have always been able to scale the variables in our equations, but for some "academic" problems there are no natural scales to use. The fundamental solution to the linear diffusion equation one such simple example. It is impossible to find reasonable time and space scales for this problem by just looking at the equation and the definition domain, $-\infty < x < \infty$ and $0 < t$. In this case it turned out not to be necessary. However, if there are no scales to use, we must combine the variables in order to obtain dimensionless equations.

We will not go into further detail on the theory of similarity solutions, where in particular, the Norwegian mathematician Sophus Lie has provided important contributions ([12], [1], [16]). However, we will, based on dimensional analysis, illustrate the method by means of an example from heat conduction. The idea of this section comes from [5].

Two infinite materials with different but constant temperatures are brought into contact at time $t = 0$, as shown in Fig. 29. We assume that heat is transported smoothly through the contact surface, and we assume one-dimensional heat conduction. The problem is to determine the temperature development in the material as time goes on. Since we assume infinite extent, and we consider the time from 0 to ∞, there is no length or time scale. In practice, the materials will have finite extent, L, and after some time T, the temperature at the ends begin to change. As long as we limit ourselves to times that are significantly smaller than T, we should be able to use the solution valid for an infinite extent of the material.

Heat transfer and storage of a material is (in its simplest form) determined by three material constants: *mass density*, ρ, $[\rho] = \mathrm{kgm}^{-3}$, *specific heat capacity*, c, $[c] = \mathrm{Jkg}^{-1}\mathrm{K}^{-1}$, and the *heat conduction coefficient*, k, $[k] = \mathrm{Js}^{-1}\ \mathrm{m}^{-1}\mathrm{K}^{-1}$.

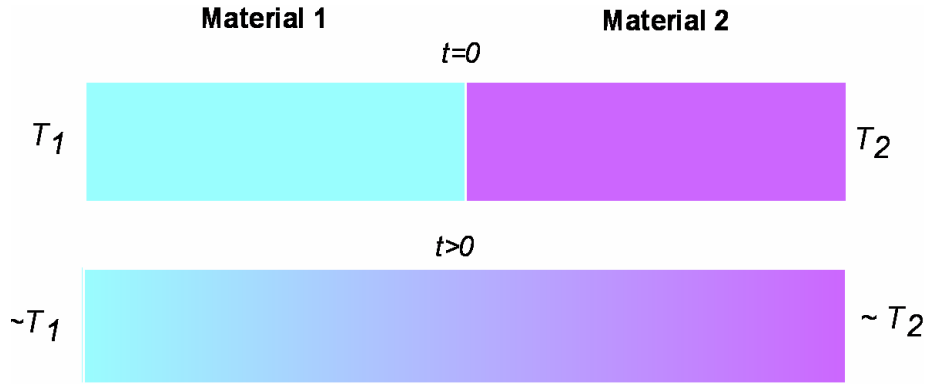

Figure 29: The temperature in the blocks for $t = 0$ and after a some time in contact.

We introduce a dimensionless temperature τ by writing

$$T = T_1 + (T_2 - T_1)\tau,$$

and quite general we expect

$$\tau = \tau(x, t, \rho_1, c_1, k_1, \rho_2, c_2, k_2). \tag{269}$$

where the indices indicate material 1 and 2. If we know the heat conduction equation, we know that ρ, c and k in the material combines into a heat diffusion coefficient $\kappa = k/c\rho$, while the heat fluxes are generally of the form $-k\nabla T$. The heat flux should be continuous across the border between the materials. This means that we must be able to simplify Eq. 269 to

$$\tau = \tau(x, t, \kappa_1, \kappa_2, k_1, k_2). \tag{270}$$

The dimension matrix for these six variables has rank 3 (check!), and we have therefore also three dimensionless combinations, for example,

$$\begin{aligned} \eta &= \frac{x}{\sqrt{\kappa_1 t}}, \\ a &= \frac{k_1}{k_2}, \\ b &= \frac{\kappa_1}{\kappa_2}. \end{aligned} \tag{271}$$

Thus, we have found that the solution of the problem must be written in the form

$$\tau\left(x,t\right) = \tau_0\left(\eta, a, b\right) = \tau_0\left(\frac{x}{\sqrt{\kappa_1 t}}, \frac{k_1}{k_2}, \frac{\kappa_1}{\kappa_2}\right). \tag{272}$$

As expected, the solution depends, on combination of x and t, and already here we make a surprising observation:

$$\tau(0, t) = \tau_0(0, a, b) \tag{273}$$

is constant and independent of time for $t > 0$! Unfortunately, dimensional analysis can not give us the exact expression for $\tau(0,t)$, but the expression for τ has reduced the problem from a partial to an ordinary diff.-equation. The heat conduction equation in material 1 now takes the form

$$\frac{\partial \tau}{\partial t} = \kappa_1 \frac{\partial^2 \tau}{\partial x^2}, \tag{274}$$

and by using $\frac{\partial \tau}{\partial t} = \frac{d\tau_0}{d\eta}\frac{\partial \eta}{\partial t}$ etc., we find

$$\frac{d^2\tau_0}{d\eta^2} + \frac{1}{2}\eta\frac{d\tau_0}{d\eta} = 0, \tag{275}$$

and similarly for material 2,

$$\frac{d^2\tau_0}{d\eta^2} + \frac{1}{2}b\eta\frac{d\tau_0}{d\eta} = 0. \tag{276}$$

Equation 276 has general solution

$$\tau_0 = A_2 + B_2 \operatorname{erf}\left(\frac{\sqrt{b}}{2}\eta\right), \tag{277}$$

whereas we for material 1 get

$$\tau_0 = A_1 + B_1 \operatorname{erf}\left(\frac{1}{2}\eta\right). \tag{278}$$

We have 4 constants from the integration, but since $\tau_0(\eta) \to 0$ when $\eta \to -\infty$, and $\tau_0(\eta) \to 1$ when $\eta \to \infty$, only two remain:

$$\begin{aligned} \tau_0 &= A\left(1 + \operatorname{erf}\left(\frac{1}{2}\eta\right)\right), \ \eta < 0, \\ \tau_0 &= 1 + B\left(\operatorname{erf}\left(\frac{\sqrt{b}}{2}\eta\right) - 1\right), \ \eta > 0. \end{aligned} \tag{279}$$

The solution found requires that $t > 0$, and then τ_0 is continuous at $\eta = 0$. This gives

$$A = 1 - B. \tag{280}$$

Finally, we use that the flux must be continuous across the contact surface,

$$k_1 \left.\frac{\partial T}{\partial x}\right|_{0-} = k_2 \left.\frac{\partial T}{\partial x}\right|_{0+}. \tag{281}$$

After introducing τ and η, this gives

$$a \left.\frac{d\tau_0}{d\eta}\right|_{0-} = \left.\frac{d\tau_0}{d\eta}\right|_{0+}, \tag{282}$$

or

$$aA\frac{1}{2} = B\frac{\sqrt{b}}{2}, \tag{283}$$

	ρ [kgm^{-3}]	c[J/kgK]	k[W/mK]	T[°C]	w[m^{2}Ks$^{\cdot 5}$]
Foot	1000	4000	0.6	37	1550
Hot coal	150	800	0.04	600	70

Table 2: Approximate values of density, specific heat and heat conduction coefficient for the human foot and burning coal

and finally,

$$A = \frac{\sqrt{b}}{\sqrt{b}+a},\ B = \frac{a}{\sqrt{b}+a}. \tag{284}$$

We get a surprisingly simple expression for the temperature of the interface:

$$T(0,t) = T_1 + (T_2 - T_1)\,A = \frac{T_1 w_1 + T_2 w_2}{w_1 + w_2}\ , \quad w_i = \sqrt{\rho_i c_i k_i}. \tag{285}$$

Now we could stop here, but the expression explains why, in winter, it feels much colder to touch a piece of metal than a piece of wood. In summer, we may get burned by a piece of metal in the sun, whereas touching a piece of wood with the same temperature is without any risk. The explanation is as follows. Your finger will have a temperature T_f and a certain w_f. Although wood and metal have the same temperature T_0, $w_{wood} < w_f$, while $w_f \ll w_{metal}$. This means that $T(0,t) \approx T_f$ when we touch the tree, while $T(0,t) \approx T_0$ when we touch the metal.

This reasoning can actually be taken even further. Some literature search has revealed the values in Table 2.

These values gives a contact temperature T_k between a *human foot* and *glowing coal* given by

$$T_k = \frac{37°\text{C} \times 1550 + 600°\text{C} \times 70}{1550 + 70} \approx 62°\text{C}. \tag{286}$$

By no means deterring! Such an explanation of why it is possible to walk on hot coals is of course rejected by a compact majority on the Internet. Some years ago, Swedish physics professor visited NTNU and let students try this astonishing experiment. He waited until the top of the coal was burned out so that it screened somewhat for exposure to the heat radiation from below. As far as I know, no one was hurt by walking over the bed.

One might wonder for how long time one could trust the similarity solution. If we assume that the outer layer of the skin (*epidermis*) is about $L = 0.5$mm, it is possible to estimate the time by setting $\eta = 1$, or

$$t = \frac{L^2}{\kappa} = \frac{\left(10^{-3}/2\right)^2 \times 1000 \times 4000}{0.6}\text{s}\ = 1.7\text{s}. \tag{287}$$

It is not worth stopping!

The temperature for this particular example is shown for 0(1)5 seconds on Fig. 30. As one can see, our estimate is realistic, but the temperature increase has reached far less than 2mm into the foot even after 5 seconds.

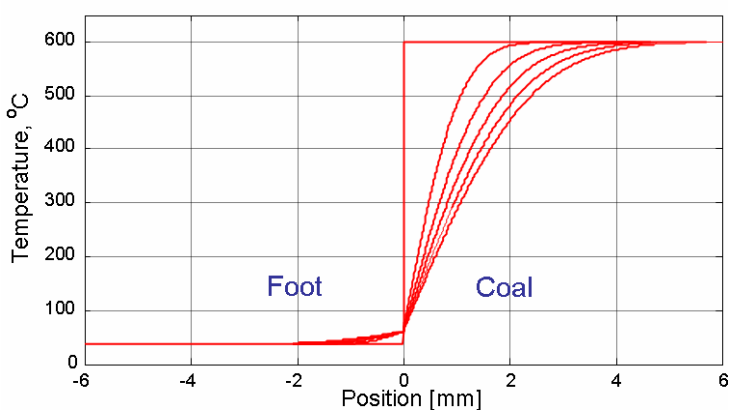

Figure 30: The temperature profile across the contact for a situation with the foot to the left and hot coals to the right shown for 0(1)5 seconds. Note that both the temperature and flux are continuous at $x = 0$ when $t > 0$.

5.5 Non-linear Diffusion

While models leading to linear diffusion equations are relatively simple to analyze, new and unexpected things when the diffusion coefficient depends on the dependent variable and the equation becomes nonlinear.

Gas flowing isothermally in a porous medium is a simple example that leads to a non-linear diffusion equation. A porous medium, such as sandstone, have small pores that gas can flow in. *Porosity*, ϕ, is the volume fraction of pores so that $0 < \phi < 1$. *Darcy's Law* says that the volume flow (m^3 / (m^2 s) is given by

$$\mathbf{q} = -\frac{K}{\mu}\nabla p \tag{288}$$

where K is a constant of proportionality called the *permeability*. Furthermore, μ is the dynamic viscosity of the gas and p the pressure. We shall assume that the state equation for the gas has the form $p = a\rho^\gamma$, where $\gamma > 0$. The conservation law for the gas will then be

$$\frac{d}{dt}\int_R \phi\rho dV + \int_{\partial R} \rho\mathbf{q}\cdot\mathbf{n}d\sigma = 0, \tag{289}$$

which gives us the differential formulation

$$\phi\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{q}) = 0. \tag{290}$$

If we insert the state equation in the expression for $\mathbf{q}$, we obtain

$$\mathbf{q} = -\frac{K}{\mu}\nabla p = -\frac{Ka\gamma}{\mu}\rho^{\gamma-1}\nabla\rho, \tag{291}$$

and therefore

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\frac{Ka\gamma}{\mu\phi}\rho^\gamma\nabla\rho\right). \tag{292}$$

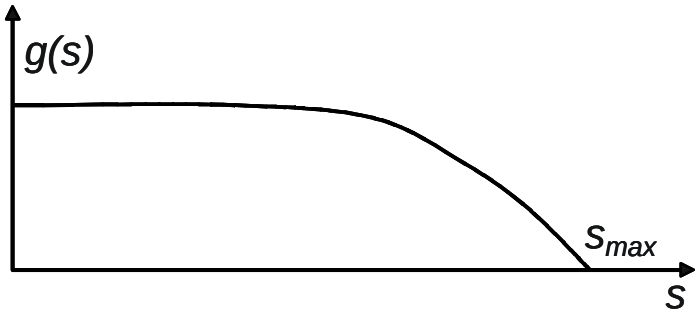

Figure 31: Sketch of the solution for $\kappa(\rho) = \rho$.

As we see, we have got a diffusion equation with a diffusion coefficient which is proportional to ρ^γ. In particular, we see that diffusion coefficient approaches 0 when the density tends to 0 (when γ is greater than 0). There is a lot of theory for non-linear diffusion equations although it is no longer possible to apply the superposition principle for solutions. Even for such equations there are similarity solutions. For equations

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\kappa(\rho)\frac{\partial \rho}{\partial x}\right) \tag{293}$$

one may look at solutions on the form $\rho(x,t) = g(s),\ s = x/t^{1/2}$. If we insert this, we obtain an ordinary differential equation for g:

$$\kappa(g)g'' + \kappa'(g)g'^2 + \frac{s}{2}g' = 0. \tag{294}$$

For $\kappa(g) = 1$ the equation is reduced to $g'' + sg'/2 = 0$ with the well-known solution

$$g(s) = A\int_{-\infty}^{s} e^{-\xi^2/4}d\xi \ + \ B. \tag{295}$$

If $\kappa(\rho) = \rho$, we obtain a solution that is sketched in Fig 31. The solution is 0 at a finite value of s, and therefore, $x_{\max} = s_{\max}\sqrt{t} \propto \sqrt{t}$. For this diffusion equation the solution spreads out at a final speed! See [4] for a more detailed analysis of diffusion.

Another class of solutions are the so-called *Barenblatt solutions* for $\kappa = \kappa_0\,(\rho/\rho_0)^m$:

$$\rho(x,t) = \rho_0\left(\frac{t_0}{t}\right)^{1/(m+2)}\left(1 - \left(\frac{x}{x_1}\right)^2\right)^{1/m}\ ,\ t > 0\ ,\ |x| \le x_1 = x_0\left(\frac{t}{t_0}\right)^{1/(m+2)} \tag{296}$$

(t_0 and x_0 may be expressed by κ_0 and $Q = \int_{x=-\infty}^{\infty} c(x,0)dx$). Contrary to the standard fundamental solution corresponding to the limit $m \to 0$, the Barenblatt solutions for $m > 0$ have finite extension on the x-axis. Examples of Barenblatt solutions for a selection of m-values are shown on Fig. 32.

Also for these equations there are theorems saying that solutions of random, but localized initial conditions approach the Barenblatt solutions when the time increases. The solutions are described in [8], but not in [1].

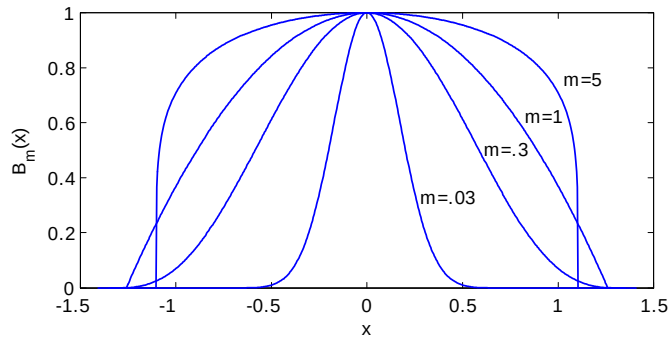

Figure 32: Barenblatt solutions shown for four different values of m. The solutions are scaled according to their maximum.

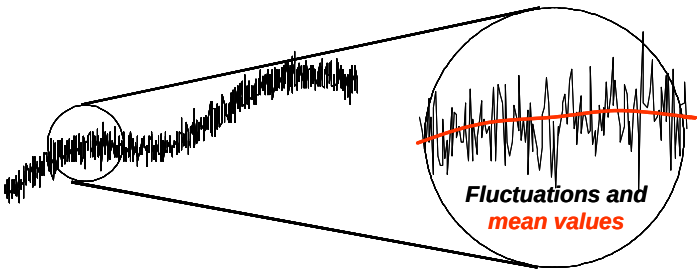

Figure 33: An observed signal consisting of rapid fluctuations and a more slowly varying mean.

6 MODELLING OF TURBULENCE

The theory of turbulence is a very good example of how we can make use of stochastic considerations in mathematical modeling. We abandon to describe the phenomenon in a deterministic way below a certain level, saying that faster variations are *stochastic* or *random.* When a complete description is out of reach, we try instead to model the evolution of the mean values, as illustrated on Fig. 33.

We shall illustrate this technique by showing how one obtains equations for turbulence in the very simplest case, namely for an incompressible, viscous fluid without any influence of external forces.

To derive the equations we need the continuity equation and Navier-Stokes equation that was derived in Sec. 4. We start by the following four differential equations

$$\nabla \cdot \mathbf{v} = 0, \tag{297}$$

$$\frac{\partial}{\partial t}\mathbf{v} + \nabla \cdot ((\mathbf{v})\mathbf{v}) = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v} \tag{298}$$

In turbulence theory a full solution of equations 297 and 298 is out of reach. The velocity

and the pressure are however seen as random variables and written

$$\begin{aligned} \mathbf{v} &= E(\mathbf{v}) + (\mathbf{v} - E(\mathbf{v}))\,, \\ p &= E(p) + (p - E(p))\,. \end{aligned} \tag{299}$$

Expectation values vary over space and time scales given by the dimensions of the phenomenon we see. These scales are called the *macroscopic scales*. "Small-scale variations" or the so-called *fluctuations* take place at the *microscopic scales*. In turbulence theory assumes that these scales are well separated, but in some situations that does not not need to be the case.

Unfortunately, the practice of turbulence theory is the opposite of the one used in statistics and probability theory: Deterministic variables are denoted by capital letters and random variables with lower case letters. In order not to confuse readers with a background in mechanics, we shall stick to this practice. Thus we use $\mathbf{V}$ for $\mathsf{E}(\mathbf{v})$, and $\mathbf{v} - \mathsf{E}(\mathbf{v})$ by $\mathbf{v}$:

$$\begin{aligned} \mathbf{v} &:= \mathbf{V} + \mathbf{v}, \\ p &:= P + p. \end{aligned} \tag{300}$$

Here $\mathbf{V}$ and P are deterministic functions of the macroscopic variables $\mathbf{x}$ and t, while $\mathbf{v}$ and p are stochastic variables with expectations 0, $\mathsf{E}\,(\mathbf{v}) = 0$, $\mathsf{E}\,(p) = 0$. Often it is assumed that $\mathbf{v}$ and p are Gaussian variables, and in general, the parameters in the probability distributions of $\mathbf{v}$ and p will depend on $\mathbf{x}$ and t.

We now introduce Eq. 300 into Eq. 297 and apply the expectation operator, which we assume commutes with the differentiation:

$$0 = E\,(\nabla \cdot (\mathbf{V} + \mathbf{v})) = \nabla \cdot \mathbf{V} + \nabla \cdot E\mathbf{v} = \nabla \cdot \mathbf{V}. \tag{301}$$

Thus we see that $\mathbf{V}$ also satisfies the Eq. 297

$$\nabla \cdot \mathbf{V} = 0 \tag{302}$$

and even $\nabla \cdot \mathbf{v} = 0$.

If Eq. 298 is treated in the same way, we obtain for component "j"

$$\rho\left(\frac{\partial V_j}{\partial t} + \nabla \cdot (V_j \mathbf{V}) + \nabla \cdot E\,(v_j \mathbf{v})\right) = -\frac{\partial P}{\partial x_j} + \nabla \cdot (\mu \nabla V_j)\,. \tag{303}$$

It is usual here to insert the tensor $\mathbf{T} = -\rho E(\mathbf{v}\mathbf{v}')$, that is, $T_{ij} = -\rho E(v_i v_j)$. This tensor is called the Reynolds stress tensor. The momentum equation for the macroscopic variables is then

$$\rho(\mathbf{V}_t + \nabla \cdot ((\mathbf{V})\mathbf{V}) = -\nabla P + \nabla \cdot (\mu \nabla \mathbf{V} + \mathbf{T}). \tag{304}$$

Eq. 304 differs from Eq. 302 by an extra force term, $\nabla \cdot \mathbf{T}$, whereas Eq. 302 is essentially the same as Eq. 297.

By subtracting Eq. 304 from 298 and inserting $\mathbf{V}+\mathbf{v}$ etc., we obtain an equation for $\mathbf{v}$. If we then take the scalar product with $\mathbf{v}$ and apply the expectation operator, the result is a transport equation for *turbulent energy per unit volume,*

$$e = \rho\frac{1}{2}E(\mathbf{v}\cdot\mathbf{v}). \tag{305}$$

The fundamental question now is how to express the Reynolds stresses by means of macroscopic variables. This is the main problem in turbulence theory. The oldest attempt was made by Boussinesq as early as 1877. He defined $\mathbf{T}$ as a function of $\partial V_i/\partial x_j$ in the same way as the stress tensor for a Newtonian fluid. Thus we can write equation

$$\rho(\mathbf{V}_t + \mathbf{V}\cdot\nabla\mathbf{V}) = -\nabla P + \nabla\cdot((\mu+\mu_T)\nabla\mathbf{V}), \tag{306}$$

and the conservation laws of mass and momentum have exactly the same form as before. The constant μ_T, with the dimension of viscosity, represents an additional "viscosity" due to turbulence, called *eddy viscosity.* It has been shown that this model matches free turbulent flow well, where μ_T can be 10 to 1000 times greater than μ. Since it is easy to apply, this has led to an extensive use as an adjustment factor in numerical computations. More advanced models relate μ_T to the local turbulent energy.

If a liquid contains a substance that is mixed with the liquid by turbulent diffusion, we will, in addition to Eq. 297 and 298 have a convection/diffusion equation for the concentration c of the substance (the "substance" may well be heat content):

$$\frac{\partial c}{\partial t} + \mathbf{v}\cdot\nabla c = \nabla\cdot(\kappa\nabla c), \tag{307}$$

where κ is the diffusion coefficient.

In the same way as for $\mathbf{V}$ and P it is possible to write the concentration $\mathsf{E}c+(c-\mathsf{E}c) := C+c$, and if this is inserted into the equation, we have

$$\frac{\partial(C+c)}{\partial t} + (\mathbf{V}+\mathbf{v})\cdot\nabla(C+c) = \nabla\cdot(\kappa\nabla(C+c)). \tag{308}$$

When we then apply the expectation operator and apply $\nabla\cdot\mathbf{v}=0$, we obtain

$$E(\mathbf{v}\cdot\nabla c) = E(\nabla\cdot(\mathbf{v}c)) - E(c\nabla\cdot\mathbf{v}) = \nabla\cdot E(\mathbf{v}c), \tag{309}$$

and

$$\frac{\partial C}{\partial t} + \mathbf{V}\cdot\nabla C = \nabla\cdot\left(\kappa\nabla C - E(\mathbf{v}c)\right). \tag{310}$$

The vector $\mathbf{q}_T = E(\mathbf{v}c)$ is called the turbulent flux. As for the Reynolds stresses, it is difficult to relate this flux to macroscopic variables. The simplest way is of course again to assume that $\mathbf{q}_T = -\kappa_T\nabla C$, as this gives the same equation as before with a new diffusion constant $\kappa+\kappa_T$.

7 PROBLEMS

7.1 Simple Problems

Problem 1

(a)

- How is *density* defined?
- How is *flux* defined?
- What is a *point source*, and how can it be described?
- What is a *distributed source*, and how can it be expressed when the flux is known?

(b) What is the flux for a substance that passively follows a fluid velocity field $\mathbf{v}(\mathbf{x},t)$? Show that the dimension of the expression is consistent with the definition of flux.

(c) How do we derive the differential form the integral form of the general conservation law?

Problem 2

A hemisphere with radius r has center at the origin, and is bounded by $0 \leq z$. The flux field $\mathbf{j}$ is defined in the space as

$$\mathbf{j}(\mathbf{x},t) = (y \sin z)\,\hat{\imath}_x + xz^3\hat{\imath}_y + z\hat{\imath}_z, \tag{311}$$

where $\{\hat{\imath}_x, \hat{\imath}_y, \hat{\imath}_z\}$ are the unit vectors along the coordinate axes. Calculate the flow of material through the curved part of the surface of the hemisphere, i.e. $|\mathbf{x}| = r$, $z > 0$. *Hint:* The solution is simple.

Problem 3

A chemical dissolved in a liquid spreads by molecular diffusion, modeled by means of the flux

$$\mathbf{j} = -\kappa \nabla \varphi, \tag{312}$$

where φ is the concentration of the substance ($\mathrm{kg/m^3}$), κ is a constant, and ∇ the gradient. What is the unit of κ? The substance decays over time, and during the time from 0 to t, the concentration (if nothing else happens) has decreased as

$$\varphi(\mathbf{x},t) = \varphi_0 e^{-t/t_0}. \tag{313}$$

This represents a distributed sink, but how can this be described as a function $q(\mathbf{x},t)$?

Use the general conservation law to derive the differential equation

$$\frac{\partial \varphi}{\partial t} = \kappa \nabla^2 \varphi - \frac{\varphi}{t_0}. \tag{314}$$

Problem 4

Find the solution surface, $z = f(x, y)$, to the quasi-linear first order partial differential equation

$$\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - 2 = 0 \tag{315}$$

so that the space curve defined by

$$\begin{aligned} x &= t, \\ y &= 1, \\ z &= t, \end{aligned} \tag{316}$$

$t \in \mathbb{R}$ is in the surface.

Hint: Follow the recipe in the Appendix of the note step-by-step.

Problem 5

How is the flux of a substance defined if it diffuses and is simultaneously transported by a moving fluid? State the conservation law both in integral and differential form.

Problem 6

(a) Define the fundamental solution $c_f(x, t)$ of the equation

$$\frac{\partial c}{\partial t} = \kappa\frac{\partial^2 c}{\partial x^2},\ x \in \mathbb{R},\ t \geq 0. \tag{317}$$

where $c(x, t)$ is the concentration of a substance and κ is a positive constant (what is κ called?).

The fundamental solution represents a unit discharge in the point $x = 0$ at time $t = 0$.

(b) Show that the total amount, $\int_{-\infty}^{\infty} c(x, t)\, dx$, remains equal to 1 for all $t > 0$. It is reasonable to consider $c(x, t)$ as a probability density on $\mathbb{R}$. What is in this case the *mean value*, η, and *standard deviation*, σ?

(c) In statistics $\pm 3 \times \sigma$ is often said to be the typical extension of a distribution. What is the $[-3\sigma, 3\sigma]$-interval here?

(d) The equation

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = \kappa\frac{\partial^2 c}{\partial x^2}, \tag{318}$$

where v is a positive constant, represents a situation where what we are considering is moving to the right with constant speed v. What is the fundamental solution in this case?

(e) Show that if the distribution at $t = t_0$ is $c(x, t_0) = h(x)$, the general solution of Eq. 1 for $t > t_0$ may be written

$$c(x, t) = h * c_f(\cdot, t - t_0)(x, t) = \int_{-\infty}^{\infty} h(s)\, c_f(x - s, t - t_0)\, ds. \tag{319}$$

(f) Use the expression in **(e)** (or a smarter way, based on the uniqueness of solutions) to find the solution for $t = 3$ when the solution of $t = 1$ is

$$h(x) = c_f(x, 1). \tag{320}$$

Can you derive a general property of convolutions of the fundamental solutions based on this?

(g) Which condition on the diffusive flux $\mathbf{j}_d$ must hold at x_0 if there is a dense wall there, so that all diffusion occurs to the right of the wall ($x \geq x_0$).

(h) Show (or argue) that if x_0 in **(g)** is larger than 0, it is possible to write the solution for a unit emission at $x = 0$ for $t = 0$ as

$$c(x,t) = c_f(x,t) + c_f(x - 2x_0, t), \ x \geq x_0, t > 0. \tag{321}$$

Problem 7

(a) Determine the fundamental solutions for

$$\frac{\partial c}{\partial t} = \kappa \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right), \ (x,y) \in \mathbb{R}^2, \ t \geq 0, \tag{322}$$

$$\frac{\partial c}{\partial t} = \kappa \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right), \ (x,y,z) \in \mathbb{R}^3, \ t \geq 0, \tag{323}$$

As long as we are interested in solutions defined on $\mathbb{R}^n$, we may use the fundamental solution to express more general solutions. The two-dimensional equation 322 may, *e.g.* describe the extent of an oil spill on the surface of the sea.

(b) Determine how a unit emission at $t = 0$ from an off shore oil platform situated at $x = 0$ will evolve if it in addition is a constant current in the sea. A variable emission over a period of time can be approximated as a number of small spills occurring at constant time intervals. State an expression for the solution in this case. The extent of the oil slick could be defined as the area where the oil film is thicker than a certain thickness, e.g. 5 molecular diameters. *Challenge*: Program and visualize this slick using Matlab or Octave.

7.2 Modeling Problems

7.2.1 The Student 10km Race

Many years ago, the students' sport club at NTNU arranged the *Student 10km Race* at the old Trondhjem Stadion. The arrangement was terribly crowded with more than 1000 participants, all starting (or trying to start) at the same time. This spectacular event, often in rain and on a terribly dirty track is the origin of the following problem.

Consider an ordinary race-track with length $L = 400$m. We assume that the mean running speed v^* decreases linearly with the density ρ^*, so that $v^* = v^*_{\max}$ for $\rho^* = 0$ students/m, and is $v^* = 0$ m/s when $\rho^* = \rho^*_{\max}$. We also consider the track to be one-dimensional and 0-shaped.

(a) State the conservation law for students (assuming no late entries or drop-outs) under these conditions, introduce dimensionless variables, and show that the differential formulation for the students' density may be written as

$$\rho_t + (1 - 2\rho)\rho_x = 0, \tag{324}$$

where $0 \leq \rho \leq 1$ and $\rho(x,t)$ is a 2π-periodic function of x.

(b) Find (in implicit form) the exact solution of Eq. 324 when

$$\rho(x,0) = \rho_0 + \varepsilon\cos(x), \quad (0 < \rho_0,\ 0 < \varepsilon < \rho_0). \tag{325}$$

Sketch the characteristics for the solution in the xt-plane and show that, as a solution of the integral conservation law, it breaks down and forms a shock in the density after some time.

(c) When will the shock start, and what happens to the shock when $t \to \infty$?

(*Hint*: Start by determining the crossing points for characteristics starting at $3\pi/2 - \theta$ and $3\pi/2 + \theta$, when θ varies from 0 to π. Try to prove that the crossing points lie on a straight line segment, and that there are no crossings elsewhere. Finally, check that the line segment is a shock and that the corresponding solution indeed satisfies the integral conservation law.

7.2.2 Two Phase Porous Media Flow

Oil is found in porous rocks. Often the porous rock is trapped between layers of solid impermeable rock, called an *oil reservoir*, and when we pump oil out, ground water enters. In so-called *enhanced oil recovery*, water is actively pumped into the reservoir and the oil is forced out. The simultaneous flow of oil and water is complex, and in order to study this in detail, small samples of rock are taken from the reservoir and investigated in the laboratory. The following model is essential in these investigations.

A long thin cylinder of porous sandstone with constant cross section A is situated along the x-axis. The pores occupy a constant fraction Φ of the volume ($0 < \Phi < 1$), and are initially filled with oil. We assume that the oil has constant density and measure the amount of oil by its volume. The sides of the cylinder are closed, but by applying a pressure at one end, it is possible to press oil or water through the stone.

In order to find an expression for the flux of oil in the x-direction , j [$\mathrm{m}^3/(\mathrm{m}^2\mathrm{s})$], we assume it only depends of the *viscosity*, μ [kg/ms], the *permeability* (inverse flow resistance) of the stone, K [m^2] , and the *pressure gradient*, $\partial p/\partial x$.

(a) Show that dimensional analysis gives

$$j = -k\frac{K}{\mu}\frac{\partial p}{\partial x}, \tag{326}$$

where k is a dimensionless constant.

Assume that the pores of the cylinder in addition to oil also contains water. All pores are either filled with water or oil, so that a volume V of rock contains a volume $S_o\Phi V$ of oil and $S_w\Phi V$ of water, where $S_o + S_w = 1$. We assume that water and oil have the same pressure and that the corresponding fluxes may be written as

$$j_i = -k_i\left(S_i\right)\frac{K}{\mu_i}\frac{\partial p}{\partial x},\ i = o, w. \tag{327}$$

The parameter $k_i\left(S_i\right)$ is called the relative permeability.

(b) Establish the conservation laws for oil and water for the part of the cylinder between $x = a$ and $x = b$. Show that if we apply a pressure gradient such that

$$q = j_o + j_w = \text{constant}, \tag{328}$$

then we have, for $S \equiv S_w$, the following hyperbolic equation for S:

$$\begin{aligned} \Phi \frac{\partial S}{\partial t} + \frac{\partial}{\partial x} f\left(S\right) = 0, \\ f\left(S\right) = \frac{q k_v(S)/\mu_v}{k_o\left(1-S\right)/\mu_o + k_v\left(S\right)/\mu_v}. \end{aligned} \tag{329}$$

(c) Assume that $\mu_o = \mu_w$, $k_o\left(1-S\right) = 1 - S^2$ and $k_w(S) = S^2$. Solve the equation (329) for $t > 0$ for a cylinder of length L when

$$\begin{aligned} S(x,0) = 1 - x/L,\ 0 \le x \le L, \\ S(0,t) = 1,\ 0 \le t. \end{aligned} \tag{330}$$

7.2.3 Reduced Speed Limit

In this problem we are considering the *standard model* for the traffic of cars along a one-way road.

(a) Describe the basis of the standard model. State the hyperbolic equation the model leads to (when no cars are assumed to enter or leave the road). When will the car density develop shocks?

(b) Between $x = 0$ and $x = 1$ there is now a reduction in the speed limit such that the maximal speed reduces to $1/2$, while the maximum density remains the same. We assume that a similar linear relation between the car velocity and the density also applies for this part of the road.

Which condition on the flux of cars has to hold at $x = 0$ and $x = 1$? Find the solution $\rho\left(x,t\right)$ for $t > 0$ and all x when

$$\rho(x,0) = \begin{cases} 1/2, & x < 1, \\ 0, & x > 1. \end{cases} \tag{331}$$

Hint: The density ρ between 0 and 1 remains constant for all $t \ge 0$.

7.2.4 Traffic Lights at a Pedestrian Crossing

In this problem we study the traffic along a one way street, and without cars entering or exiting in the first part. All variables are scaled so that the car density ρ is between 0 and 1, and car velocity v is equal $1 - \rho$.

(a) Show how to find an expression for the shock velocity U of a jump in car density, and derive that in this case, $U = 1 - \rho_1 - \rho_2$, where ρ_1 and ρ_2 are the densities on each side of the shock.

For $t < 0$, there is a constant car density $\rho = 1/2$ on the street. Between $t = 0$ and $t = 1$ the cars face a *red* light at a pedestrian crossing at $x = 0$. For $t > 1$, the light is again *green*.

(b) Determine the solution $\rho(x, t)$ for $t \geq 0$.

(*Hint*: Make a sketch of the situation in an x/t-diagram. Show that the solution for ρ has to be found in five different domains, of which the values in four of them are obvious. In order to determine the domains it is necessary to determine their exact borders).

At another place on the street, a second one way street of the same type as the first merges with the first street.

(c) Which condition must hold at the junction? Assume that the flux on the first street towards the junction is constant, $j_1 = 1/8$, and the corresponding car density is less than $1/2$. Describe the development of car density on the streets when the density ρ_2 on the second street increases from 0 to 1. The drivers on the first street have the right of way, but are flexible and let cars enter from the second street if this is possible. In particular, look at what happens when the flux on the second street reaches $1/8$.

7.2.5 A Water Cleaning System

A part of a water cleaning system is modelled as a tube of length L along the x^*-axis, where polluted water flows with constant velocity V. The tube also contains absorbers that remove the pollution. The concentration of pollutant in the water is c^*, measured as amount per length unit of pipe. Similarly, the amount of absorbed pollutant per length unit of pipe is denoted ρ^*. The maximum value of ρ^* is A. Some of the absorbed pollutant will over time re-enter the water stream. The absorption and re-entering is modelled by the equation

$$\frac{\partial \rho^*}{\partial t^*} = k_1 (A - \rho^*) c^* - k_2 \rho^*. \tag{332}$$

(a) State the integral conservation law for the pollutant and show that this leads to the differential form

$$\frac{\partial}{\partial t^*} (c^* + \rho^*) + \frac{\partial}{\partial x^*} (V c^*) = 0. \tag{333}$$

Based on the integral law, establish that a discontinuity in the concentrations, moving with velocity U^*, has to fulfil

$$U^* = \frac{c_2^* - c_1^*}{(c_2^* + \rho_2^*) - (c_1^* + \rho_1^*)} V, \tag{334}$$

where (c_1^*, ρ_1^*) and (c_2^*, ρ_2^*) are the concentrations on each side of the discontinuity.

(b) Introduce suitable scales and show that the differential equations may be written

$$\frac{\partial}{\partial t}(c+\rho)+\frac{\partial c}{\partial x}=0, \tag{335}$$

$$\varepsilon\frac{\partial \rho}{\partial t}=(1-\rho)\,c-\beta\rho. \tag{336}$$

Explain the meaning of ε and β (*Hint*: Use the same scale for ρ^* and c^*).

Assume that the tube is infinitely long in both directions and consider analytic solutions of equations 335 and 336 in the form of "fronts", travelling with velocity U, that is,

$$c(x,t)=C(x-Ut), \tag{337}$$

$$\rho(x,t)=R(x-Ut). \tag{338}$$

With $\eta=x-Ut$, we limit ourselves to the special case where $C(\eta)$ and $R(\eta)$ satisfy

$$\lim_{\eta\to-\infty}C(\eta)=1, \tag{339}$$

$$\lim_{\eta\to\infty}C(\eta)=0, \tag{340}$$

$$\lim_{\eta\to-\infty}R(\eta)=\frac{1}{1+\beta}, \tag{341}$$

$$\lim_{\eta\to\infty}R(\eta)=0. \tag{342}$$

(c) Insert 337 and 338 into Eq. 335, integrate once, and use the behaviour at $-\infty$ and ∞ to determine U and a simple relation between C and R. Use this information and Eq. 336 to determine $C(\eta)$ and $R(\eta)$. How is the behaviour of the solution when $\varepsilon\to 0$? (*Hint*: The equation

$$\frac{dy}{d\zeta}=y\left(-1+\frac{y}{M}\right) \tag{343}$$

has a solution

$$y(\zeta)=M\frac{1}{1+\exp\zeta} \tag{344}$$

for $0<y<M$).

(d) Assume $\varepsilon=0$ in Eq. 336 so that the system (335 and 336) simplifies to

$$\rho=\frac{c}{c+\beta},\ \frac{\partial Q(c)}{\partial t}+\frac{\partial c}{\partial x}=0, \tag{345}$$

$$Q(c)=c+\frac{c}{c+\beta},\ -\infty<x<\infty,\ t\geq 0. \tag{346}$$

Consider the initial condition

$$c(x,0)=\begin{cases}1, & x<0,\\ 0, & x>0.\end{cases} \tag{347}$$

Show that the corresponding solution of Eq. 345 develops a shock. Determine the shock velocity from the expression in point (a) and compare to the result in (c).

7.2.6 River Contamination

Discharge of contaminants into a river will be transported with the flow (*convection*) and spread due to turbulence mixing and varying water velocity (*diffusion*). Consider a one-dimensional river with mean flow U and the diffusion coefficient κ.

(a) Derive the expression for the flux of contaminants under these simple conditions, and find a length scale of the extent of an instantaneous point discharge after this has been carried a length L down the river by means of the current. At the point $x = 0$ there is a continuous discharge of a substance A so that the concentration in the river becomes $a(x,t)$. The substance A is converted into substance B with constant rate μ. Thus, for a water sample from the river we would have

$$\frac{da}{dt} = -\mu a. \tag{348}$$

The substance B decays with rate λ, and for the same water sample, the concentration $b(x,t)$ of B fulfils

$$\frac{db}{dt} = +\mu a - \lambda b.$$

(b) State the conservation laws for A and B on the integral and differential form.

(c) The discharge at $x = 0$ takes place at a constant rate q_0 (amount per time unit). Neglect diffusion and decide how far down the river the concentration of the substance A is at its highest when we assume that $\lambda = \mu$.

Hint: The differential equation $\frac{dy}{dt} + ky = e^{-kt}$ has the general solution $y(t) = C_1 e^{-kt} + te^{-kt}$.

7.2.7 Lake Sedimentation

A river flows into a lake. The river brings sand and clay so that the lake is filled up over time. We shall formulate and analyze a simple one-dimensional model for how the lake is filled, and assume that it reaches from $x = 0$ to $+\infty$ and has a constant depth h at $t = 0$. Conditions across (in the y-direction) are assumed to be constant.

The amount of sand and clay which settle on the bottom per time and area unit is $q(x,t)$. We write the depth $z = b(x,t)$, $x \geq 0$, $t \geq 0$, and assume $b(x,t) \leq 0$. If the bottom tilts (is not horizontal), the particles on the bottom will continue to move, and it has been found that the mass flux is proportional to the slope, that is, the volume flux may be written

$$j = -k\frac{\partial b}{\partial x}. \tag{349}$$

(a) Write the conservation equation in integral form for a part of the bottom , $x_0 \leq x \leq x_1$, and show that the differential form is identical to the heat diffusion equation,

$$\frac{\partial b}{\partial t} = k\frac{\partial^2 b}{\partial x^2} + q. \tag{350}$$

(b) Assume that *all* sand and clay enter at $x = 0$ (i.e. $q = 0$ for $x > 0$), and that the amount entering is always sufficient for Eq. 350 to hold for $t > 0$. Argue that the solution to Eq. 350 will be a similarity solution in this case, and find $b(x,t)$ for $x \geq 0$ and $t > 0$. (*Hint:*The equation

$$\frac{d^2y}{d\eta^2} + \frac{\eta}{2}\frac{dy}{d\eta} = 0 \tag{351}$$

has the general solution $A + B\,\mathrm{erf}\,(\eta/2)$, where $\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-s^2)\,ds$).

(c) A more realistic scenario is that the shore, $s(t)$, moves forward into the lake over time. Assume that a constant volume of sand and clay enter the basin per time unit, q_0, and that all sand and clay enter at the shore.

The solution will then have a stationary shape and may be written by means of a function b_0 so that

$$b(x,t) = \begin{cases} 0 & x \leq s(t) = Ut + x_0 \\ b_0(x - Ut - x_0) & x > Ut + x_0 \end{cases} \tag{352}$$

Determine the velocity U and the solution in this case.

7.2.8 The Insect Swarm

Flying insects sometimes form dense swarms where the insects are attracted to each other. On the other hand, the swarm has a certain extension, which implies that there is also something preventing the insects from coming too close to each other. This modeling study tries to explain this as a balance between the attraction towards the swarm and a random motion modelled as a diffusion.

The model is for simplicity one-dimensional, where the insects are assumed to stay in a straight tube (For more information, see: P. Grindrod: *Patterns and Waves*, Claredon Press, Oxford, 1991, pp. 188–189).

An insect swarm with density $\rho(x^*, t^*)$ is situated in a long tube parallel to the x^*-axis. In the swarm *random flight* (diffusion) contributes to spreading the insects, while the insects in the swarm are also attracted towards the center of the swarm. This latter effect can be modeled as a mean drift velocity w,

$$w(x^*, t^*) = -K\left(\int_{-\infty}^{x^*} \rho(s^*, t^*)\; ds^* - \int_{x^*}^{\infty} \rho(s^*, t^*)\; ds^*\right). \tag{353}$$

We shall assume that the total amount of insects,

$$M = \int_{-\infty}^{\infty} \rho(x^*, t^*)\; dx^*, \tag{354}$$

remains constant.

(a) Explain why the model for w is not unreasonable, and state the conservation law for insects in integral form.

(b) Introduce the cumulative distribution of insects,

$$v^*(x^*, t^*) = \int_{-\infty}^{x^*} \rho(s^*, t^*)\ ds^*, \tag{355}$$

and show from the conservation law that v^* satisfies the equation

$$\frac{\partial v^*}{\partial t^*} = \sigma \frac{\partial^2 v^*}{\partial x^{*2}} - K(M - 2v^*)\frac{\partial v^*}{\partial x^*} \tag{356}$$

where σ is the diffusion coefficient. We assume that σ and K are constant.

(c) For a swarm with a diameter L there are two characteristic time scales,

$$T_K = \frac{L}{KM} \quad \text{and} \quad T_D = \frac{L^2}{\sigma} \tag{357}$$

What do these scales signify? Scale the equation for v^* when $x^* = \mathcal{O}(L)$, $t^* = \mathcal{O}(T_K)$ and $T_D \gg T_K$, and show that it can be stated as

$$\frac{\partial v}{\partial t} = \varepsilon \frac{\partial^2 v}{\partial x^2} - (1 - 2v)\frac{\partial v}{\partial x}. \tag{358}$$

What is the interpretation of ε?

(**d**) Determine $\rho^*(x^*, t^*)$ when

$$\rho^*(x^*, 0) = \begin{cases} \frac{M}{2L}, & |x^*| < L \\ 0, & \text{otherwise} \end{cases}, \tag{359}$$

when we ignore the effect of diffusion. How do you expect, in rough terms, that the exact solution looks when diffusion is included?

(**e**) To examine the solution in (**d**) after a long time, it is reasonable to consider a length scale $L' = \sigma/(KM)$ and a time scale $T \gg T_K$ and T_D. Show that with this scaling the equation for v is independent of t to leading order. Verify that the leading order equation has a solution

$$\rho^*(x^*, t^*) = A\frac{1}{\cosh^2(Bx^*)} \tag{360}$$

and determine A and B for a solution with the initial distribution as in (**d**).

Hint: Use that $\frac{d}{dx}\tanh(x) = \cosh^{-2}(x)$, and try a v with the correct behaviour when $x \to -\infty$ and $x \to \infty$.

8 APPENDIX: FIRST ORDER QUASI-LINEAR PDES – A "FIRST AID COURSE"

This appendix gives a short introduction to the solution of first order partial differential equations (PDEs) that occur in connection with models based on conservation principles. It aims at students only with Calculus background.

8.1 What Does the Title Mean?

The title contains several words which may be unknown:

- *First Order* = only first order derivatives occur in the equation
- *Quasi-linear* = the equation is linear in the first order derivatives
- *Partial* = there is more than one independent variable

8.2 Equations and Solutions

The theory below is illustrated for one variable z dependent on two independent variables x and y. Equations with more independent variables are solved in a similar way.

PDEs are divided into several classes, and for an equation to belong to the class in the title, it is necessary that it can be put into what we call the *normal form*. This means that the equation can be written

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}-R(x,y,z)=0. \tag{361}$$

Here P, Q and R are functions only of x, y, and z, and do not contain any derivatives. Note that $\partial z/\partial x$ and $\partial z/\partial y$ only occur in the first power, but there is no such limitation for z in P, Q and R. The reason for the minus in front of the third term will become clear below.

A *solution* of Eq. 361 is a function,

$$z=f(x,y), \tag{362}$$

which satisfies the equation:

$$P(x,y,f(x,y))\frac{\partial f(x,y)}{\partial x}+Q(x,y,f(x,y))\frac{\partial f(x,y)}{\partial y}-R(x,y,f(x,y))=0. \tag{363}$$

If we consider a regular coordinate system, $(x,y,z)\in\mathbb{R}^3$, the function $z=f(x,y)$ will define a *surface* in $\mathbb{R}^3$. Typically, finding a solution to Eq. 361 means to find a function $f(x,y)$ fulfilling some addition conditions, e.g. having given values on some curve in the xy-plane. It will soon become clear that solving a PDE is radically different from

solving ordinary diff. equations, although ordinary equations sometimes come up during the solution process.

From Calculus we remember that the vector

$$\mathbf{n} = \left[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}, -1\right] \tag{364}$$

is a *normal vector* (perpendicular) to the surface $z = f(x,y)$ in the point (x,y,z) (Try to derive this yourself if you do not know it).

A *vector field* $\mathbf{V}(x,y,z)$ in $\mathbb{R}^3$ is defined in terms of three functions making up the three components of the vector, say

$$\mathbf{V}(x,y,z) = [P(x,y,z), Q(x,y,z), R(x,y,z)]. \tag{365}$$

A vector field defines a set of *stream lines* in space. Curves in space may be parametrized by a variable s and written as

$$\mathbf{r}(s) = [x(s), y(s), z(s)], s \in \mathbb{R}. \tag{366}$$

The stream lines for the vector field $\mathbf{V}$ satisfy the following system of differential equations

$$\frac{d\mathbf{r}}{ds} = \mathbf{V}(x,y,z), \tag{367}$$

or, written out,

$$\begin{aligned} \frac{dx}{ds} &= P(x,y,z), \\ \frac{dy}{ds} &= Q(x,y,z), \\ \frac{dz}{ds} &= R(x,y,z). \end{aligned} \tag{368}$$

In general, one can set $\mathbf{r}(s_0) = \mathbf{r}_0 = [x_0, y_0, z_0]$ and solve the system in equations 368 in order to find the stream line through $\mathbf{r}_0$. In the PDE literature, you often find Eq. 368 written as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{369}$$

This means *exactly the same* and is nothing but a short way of writing Eq. 368.

We now make an interesting observation:

- *The normal vector to a solution of Eq. 361 is perpendicular to the stream lines of the vector field* $\mathbf{V}$*, defined as in Eq. 365 with* P*,* Q*, and* R *from Eq. 361.*

This is quite obvious since

$$\mathbf{V} \cdot \mathbf{n} = [P,Q,R] \cdot \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right] = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} - R = 0! \tag{370}$$

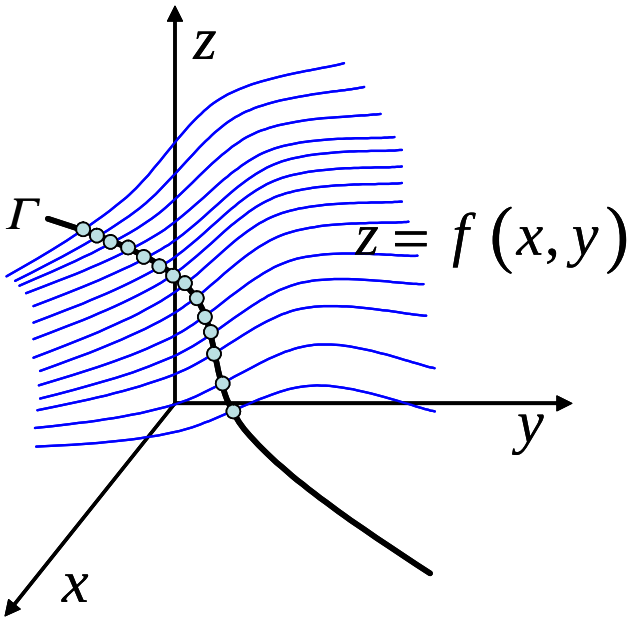

Figure 34: The characteristics through Γ define a solution to the Cauchy problem.

In the PDE theory, the stream lines are called *characteristic curves*, or simply *characteristics*.

It is the important observation above that now makes it possible for us to solve the so-called *Cauchy problem*:

Given a curve Γ *in space. Find a function* $z = f(x, y)$ *that satisfies Eq. 361 and is such that the curve* Γ *is contained in the surface defined by the solution.*

The Cauchy problem is the common name for such problems, resembling what we would call an initial value problem for ordinary diff. equations.

If we think in practical terms, it is actually not so difficult to imagine how this could be carried out: For all points on Γ, we find the characteristic curves through the points. When we then move along Γ, the characteristics slice out a surface in space. By the way the surface is made, the normal vectors to the surface must be orthogonal to the characteristics. In other words, we have actually got the situation in Eq. 370, and have found a solution to Eq. 361, as illustrated in Fig. 34.

When P, Q, and R are nice and reasonable functions, the solution of Eq. 368 will be unique. This means that only one characteristic curve can pass through each point in space, or that two different characteristic curves in space can never collide. However, this does not prevent a solution, as the one on Fig. 34, from "folding over", meaning that one has two different z-values for each point (x, y). We shall see later that this complicates matters for real world problems where z has a physical meaning.

There are also two special situations where the characteristic method needs to be modified. The first situation is when Γ itself (or parts of it) is a characteristic curve. When this

is the case, a *unique* solution to the equation can not be obtained. The second possibility is when there exists functions $f(x,y)$ so that

$$\begin{aligned} P(x,y,f(x,y)) &= 0, \\ Q(x,y,f(x,y)) &= 0, \\ R(x,y,f(x,y)) &= 0. \end{aligned} \tag{371}$$

Such functions are called *singular solutions*, since they obviously satisfy Eq. 361.

Even if this way of solving the equations may seem simple, it is quite implicit and not always so easy to carry out. Finally, even if both Γ and the characteristics are known, it may be difficult or even impossible to write the solution in the explicit form $z = f(x,y)$.

8.3 An Example

The following example is taken from Volume III of the classic calculus textbook "*Lærebok i matematisk analyse*" by R. Tambs Lyche, §282 (used at NTH for generations).

We are expected to find the solution of the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - 1 = 0 \tag{372}$$

passing through the space curve

$$\mathbf{r}(t) = -\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k},\ t \in \mathbb{R} \tag{373}$$

($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the respective axes).

The first step will be to determine the characteristics, that is, to solve

$$\begin{aligned} \frac{dx}{ds} &= P(x,y,z) = x, \\ \frac{dy}{ds} &= Q(x,y,z) = y, \\ \frac{dz}{ds} &= R(x,y,z) = 1. \end{aligned} \tag{374}$$

It is not difficult to solve these equations since they do not interfere with each other:

$$\begin{aligned} x(s) &= C_1 e^s, \\ y(s) &= C_2 e^s, \\ z(s) &= s + C_3. \end{aligned} \tag{375}$$

It remains to find expressions for the special characteristics that pass through the space curve, and thus eliminate the free constants C_1, C_2 and C_3 in Eq. 375. There are several ways to proceed, and the following is somewhat simpler than the method used in the

reference. Let us (without loss of generality) assume that the characteristics cross the curve for $s = 0$. We then obtain

$$\begin{aligned} x(0) &= C_1 = -1, \\ y(0) &= C_2 = 2t, \\ z(0) &= C_3 = t^2. \end{aligned}$$

The solution is thus the surface defined in *parametric* form as

$$\begin{aligned} x &= C_1 e^s = -e^s, \\ y &= C_2 e^s = 2te^s, \\ z &= s + C_3 = s + t^2, \end{aligned} \tag{376}$$

for the pair of parameters $(s, t) \in \mathbb{R}^2$.

In this special case it is actually also possible to eliminate s and t, and write z as a function of x and y. From the first two equations in Eq. 376 we see that

$$s = \ln(-x), \tag{377}$$

$$t = -\frac{y}{2x}. \tag{378}$$

If this is inserted into the third equation we obtain

$$z = \ln(-x) + \left(\frac{y}{2x}\right)^2. \tag{379}$$

8.4 Recipe

1. Be sure that the PDE is written in the form

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} - R(x, y, z) = 0, \tag{380}$$

   with no derivatives in P, Q, and R. Do not forget the minus sign in front of R!

2. Determine the Γ-curve and put it in the parametric form

$$\mathbf{r}(s) = [x(s), y(s), z(s)], s \in \mathbb{R}.$$

   Often, it is possible to use x or y as the parameter, say, $x = x$, $y = g(x)$, $z = f(x)$.

3. Form the ODE system for the characteristic curves,

$$\begin{aligned} \frac{dx}{ds} &= P(x, y, z), \\ \frac{dy}{ds} &= Q(x, y, z), \\ \frac{dz}{ds} &= R(x, y, z), \end{aligned} \tag{381}$$

   and solve it by some standard method (In real life, this may have to be carried out numerically).

4. Determine the collection of characteristics passing through Γ by imposing appropriate initial conditions on the solution found in the previous point. Make sure this defines a surface, which, as in the example, may sometimes be reduced to the explicit form $z = f(x, y)$.

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