INTRODUCTION TO FINITE VOLUME METHODS

This is a *very short* introduction to finite volume methods. Finite volume methods are popular because of their flexibility and the fact that they are based on the same simple physical principle as the equations they aim to approximate. Let us consider three types of equations. The first type is a hyperbolic conservation law:

$$\frac{\partial u}{\partial t} + \nabla \cdot (F(u)) = 0, \tag{1}$$

where the flux $F(u)$ is a function of the unknown u. The second type is a diffusion equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u), \tag{2}$$

where the diffusion coefficient D depend on x. We can rewrite (2) in the same form as (1) but, in this case, the flux F depends linearly on the gradient,

$$F = -D \nabla u. \tag{3}$$

The third type is an elliptic equation which corresponds to the diffusion equation at equilibrium,

$$\nabla \cdot (D \nabla u) = 0. \tag{4}$$

Let us start by setting up a discretization for a flux problem where, given a source term, we try to compute the corresponding flux based on a conservation principle. We look at the equation

$$\nabla \cdot F = q. \tag{5}$$

After integrating (5) over an arbitrary domain Ω, we obtain

$$\int_\Omega q(x)\, dx - \int_{\partial\Omega} F \cdot n \, dS = 0, \tag{6}$$

which - in words - means that, for the amount of the quantity we are considering here (it can be many things: mass, heat, momentum ...), we have

$$\left\{ \begin{array}{c} \text{Local} \\ \text{production} \\ \text{in } \Omega \end{array} \right\} - \left\{ \begin{array}{c} \text{What} \\ \text{comes out} \end{array} \right\} + \left\{ \begin{array}{c} \text{What} \\ \text{comes in} \end{array} \right\} = 0 \tag{7}$$

Now, let us proceed with a discrete analog. We consider an unstructured mesh, that is, a mesh where the cells are polygons with an arbitrary number of faces. In Figure 1, we plot two neighboring cells and $n_{i,j}$ denotes the exterior (with respect to cell Ω_i) normal of the face between cells Ω_i and Ω_j. For the cell Ω_i, we introduce the average source term q_i, defined as

$$q_i = \int_{\Omega_i} q(x)\, dx.$$

Integrating (5), as we did to obtain (6), over the domain Ω_i, we obtain

$$\sum_{j \in N(i)} \int_{\partial\Omega_{i,j}} F \cdot n_{i,j} \, dS = q_i. \tag{8}$$

Here, $N(i)$ denotes the index of the neighboring cells of Ω_i and $\partial\Omega_{i,j}$ denotes the intersection between Ω_i and Ω_j. Now, we have to define a way to approximate

Date: October 30, 2013.

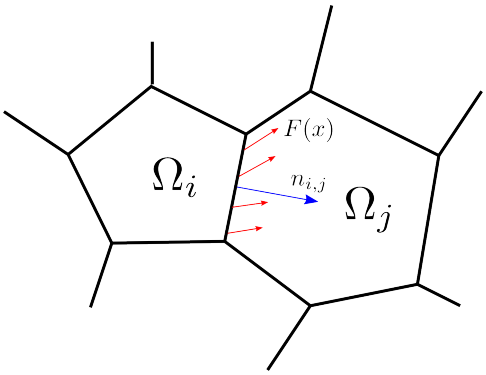

FIGURE 1. Two neighboring cells of an unstructured mesh.

the integrals in (8). This will be done in different ways but we end up with an numerical flux $F_{i,j}$ on the face $\partial\Omega i, j$, which approximate the integrated flux on the face, namely,

$$F_{i,j} \approx \int_{\partial\Omega_{i,j}} F \cdot n_{i,j}\, dS. \tag{9}$$

After rewriting (8), we obtain

$$\sum_{j \in N(i)} F_{i,j} = q_i. \tag{10}$$

For (10), the summary given by (7) is as relevant as before. Let us apply the method to the elliptic equation (4). We decompose the equation in two,

$$\begin{cases} \nabla \cdot F = 0 \\ \quad F = -D\nabla u. \end{cases} \tag{11}$$

The discretization of the first equation in (11) follows from (10) and we have

$$\sum_{j \in N(i)} F_{i,j} = 0. \tag{12}$$

We have to choose an approximation for the numerical fluxes $F_{i,j}$, as defined by (9), which in this case is

$$F_{i,j} \approx -\int_{\partial\Omega_{i,j}} D\nabla u \cdot n_{i,j}\, dS. \tag{13}$$

In reservoir simulation, it is common to use a two point flux approximation (TPFA), which is the simplest approximation one can consider. This approximation is given by a single interface coefficient depending on the permeability (which corresponds to D in the notation introduced here) and on the geometry. This coefficient is called the *transmissibility* and is defined such that

$$F_{i,j} = -T_{i,j}(u_j - u_i) \approx -\int_{\partial\Omega_{i,j}} D\nabla u \cdot n\, dS \tag{14}$$

Here, u_i and u_j are average values on cells, that is,

$$u_i = \frac{1}{V(\Omega_i)} \int_{\Omega_i} u\, dx, \tag{15}$$

where $V(\Omega_i)$ denotes the volume of the cell Ω_i. The numerical scheme defined by

$$\begin{cases} \sum_{j\in N(i)} F_{i,j} = 0 \\ F_{i,j} = -T_{i,j}(u_j - u_i), \end{cases} \tag{16}$$

which we can rewrite in the compact form

$$-\sum_{j\in N(i)} T_{i,j}(u_j - u_i) = 0 \tag{17}$$

is thus a finite volume approximation of the elliptic equation (4). To solve the diffusion equation (2), we have to discretize in time. We consider a time discretization parameter Δt and set $t^n = n\Delta t$. First we integrate (2) over a cell Ω_i and obtain

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\, dx = \sum_{j\in N(i)} F_{i,j}, \tag{18}$$

where $F_{i,j}$ denote the numerical fluxes as introduced earlier. Then, we make a first order approximation of the left-hand side in (18),

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\, dx \approx V(\Omega_i)\frac{u_i^{n+1} - u_i^n}{\Delta t}, \tag{19}$$

where

$$u_i^n = \int_{\Omega_i} u(t^n, x)\, dx. \tag{20}$$

For stability reason, which we do not develop here, the right-hand side in (18) has to be evaluated at time $n+1$ (we need an implicit scheme). Then using, the TPFA approximation, we get the following scheme

$$\begin{cases} u_i^{n+1} - u_i^n = \dfrac{\Delta t}{V(\Omega_i)} \displaystyle\sum_{j\in N(i)} F_{i,j}^{n+1}, \\ F_{i,j}^{n+1} = -T_{i,j}(u_i^{n+1} - u_j^n). \end{cases} \tag{21}$$

A finite different volume formulation for the conservation law (1) can be derived using the same principle. The difficulty in this case is that the solutions of (1) are typically discontinuous and the choice of the numerical flux becomes a highly nontrivial question.