TMA4195 - MATHEMATICAL MODELING (FALL 2013).

PROJECT DESCRIPTION.

SOME QUANTITATIVE MODELS IN OIL RECOVERY.

Introduction

In this project, we want to set up simple models which can give us quantitative results about the profitability of reservoirs, depending on their physical properties (both for the rock and the oil) and of the initial configuration.

1. Primary recovery

Reservoirs are porous media. The amount of pores in the media determines the porosity which is the fraction of volume that fluids can occupy inside the material. The pores are connected so that the fluids may move through the media. The resistance that the rock opposes to the fluid flow is characterized by the permeability, a parameter which is introduced later in the text. As a result of geological processes, pressure builds up in reservoirs. The rock is compressed and the volume of oil contained in the rock increases. Then, a significant part of the oil can be recovered by simply drilling a well in the formation. When the well is opened, the pressure decreases in the reservoir, the rock expands and presses the oil out of the reservoir through the well. This phase of oil recovery is called primary recovery. The amount of oil that can be extracted in this way depends on the compressibility of the rock, which relates the density of the rock with pressure, and the initial pressure in the reservoir.

Let us denote by ϕ the porosity of the rock. For a compressible rock, ϕ depends on the pressure. If we denote by c_r the rock compressibility, we have

$$\frac{1}{\phi}\frac{d\phi}{dp} = c_r.$$

Let $p_{b,\text{initial}}$ be the initial pressure at the bottom of the well, before the well has been opened, and p_b denote the pressure at the bottom of the well at the end of the primary recovery when hydrostatic equilibrium is reached inside the whole reservoir so that we have

$$p_b = p_{\text{atmospheric}} + \rho_o g(z_b - z_0), \tag{1.1}$$

see Figure 1. Here, ρ_o denotes the density of the oil, z_b the depth of the well and z_0 the reference depth where the pressure is equal to $p_{\text{atmospheric}}$.

Question 1a. *We assume that the reservoir is initially completely filled with oil. Compute the fraction of the oil that can be recovered by primary recovery and show that*

$$\frac{\textit{Amount of recovered oil}}{\textit{Total initial amount of oil in reservoir}} = 1 - e^{-c_r(p_{b,initial} - p_b)}. \tag{1.2}$$

Date: October 30, 2013.

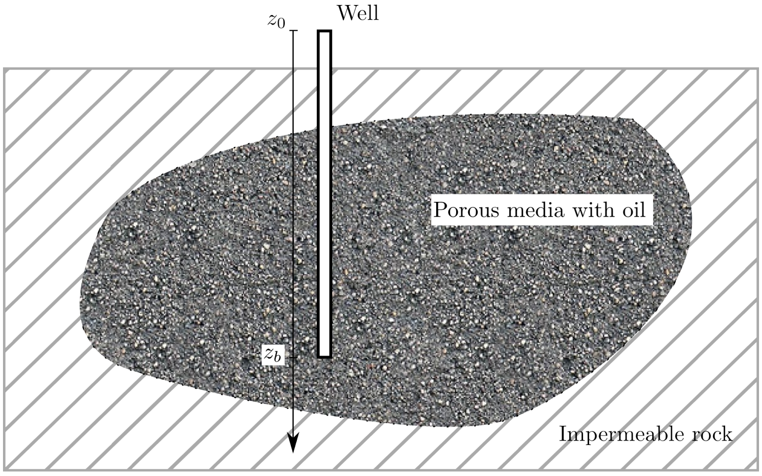


FIGURE 1. Reservoir description

We want now to study the evolution of the production rate in the well during primary recovery. The rock is a porous media and the volume flux of the flow is described by Darcy's law:

$$u = -\frac{1}{\mu}\mathbf{K}(\nabla p + \rho_o g \boldsymbol{e}_z). \tag{1.3}$$

Here, u is the volume flux per unit area, μ is the viscosity, $\mathbf{K}$ is the permeability matrix, g the gravity constant and $\boldsymbol{e}_z = [0, 0, 1]^t$. The permeability matrix depends on the type of the rock and, therefore, it is usually a function of the spatial coordinates, that is, $\mathbf{K} = \mathbf{K}(x, y, z)$.

Question 1b. *Show that the equation corresponding to the mass conservation of oil is given by*

$$\frac{\partial \phi(p)}{\partial t} - \nabla \cdot (\frac{1}{\mu}\mathbf{K}(x, y, z)(\nabla p + \rho_o g \boldsymbol{e}_z)) = 0 \tag{1.4}$$

This equation is of parabolic type.

Question 1c (Open). *We want to compute the fraction of recovered oil as a function of time to see how fast the limiting value given in* (1.2) *is reached. In this aim, simplify* (1.4) *as much as you need in order to obtain an equation in one dimension which you can solve, using for example the method of separation of variables. Let us first assume that the well is located at one end of the interval domain. On the other end, we impose no flux boundary conditions. To simplify the equation, you may use some of the following assumptions*

- *Assume that the porosity function $\phi(p)$ is an affine function of pressure.*
- *Simple geometry of the reservoir.*
- *Symmetries in the horizontal plane (cylindrical invariance or translation invariance in the x and y direction)*
- *Hydrostatic equilibrium in the vertical direction, that is,*

$$p(t, x, y, z) = p(t, x, y, z_b) + \rho_o g(z - z_b)$$

- *Constant rock permeability, that is, $\mathbf{K}(x, y, z) = \mathbf{K}$, for a given constant matrix $\mathbf{K}$.*
- *Scalar permeability matrix, that is, $\mathbf{K} = k\,\mathrm{Id}$, for $k \in \mathbb{R}$.*

What is the effect of moving the well to the middle of the interval? What about adding more wells?

Question 1d (Open). *In several space dimensions, how do we model the well? What are the boundary conditions? Find a numerical scheme which can solve Equation* (1.4) *and implement the scheme.*

2. Secondary Recovery

Secondary recovery consists of injecting water into the reservoir to restore a pressure gradient which can displace the oil which is left after primary recovery. One of the issue here is that oil and water have different mobilities. Morally speaking, because water is less viscous, it *travels faster* than oil. The consequence is that we end up by producing in one end the water we are injecting in the other end. We want to set up equations which enable us to understand and quantify this phenomenon. We have two phases, water and oil. For a given phase α ($\alpha = w$ for water and $\alpha = o$ for oil), the saturation s_α represents the fraction of the pore volume occupied by the phase α. Since water and oil fill up the whole pore volume, we have $s_w + s_o = 1$. Again we use Darcy law, which holds for each phase $\alpha \in \{w, o\}$:

$$u_\alpha = -\frac{s_\alpha}{\mu_\alpha} k \nabla p. \tag{2.1}$$

Here, we neglect gravity and consider a constant scalar permeability. Again, mass conservation of water and oil give us the governing equations for the system,

$$\frac{\partial s_\alpha \phi \rho_\alpha}{\partial t} - \nabla \cdot \big(\frac{\rho_\alpha s_\alpha}{\mu_\alpha} k \nabla p\big) = 0, \tag{2.2}$$

for $\alpha \in \{w, o\}$

Question 2a. *Explain the presence of the term s_α in* (2.1) *and derive the mass conservation equations* (2.2).

We simplify the equation by considering the one dimensional case. We assume that the rock and the liquids are incompressible, that is, the porosity ϕ and the densities ρ_w, ρ_o are constant. Then, we obtain

$$\phi \frac{\partial s_w}{\partial t} - \frac{\partial}{\partial x}\big(\frac{k s_w}{\mu_w}\frac{\partial p}{\partial x}\big) = 0, \tag{2.3a}$$

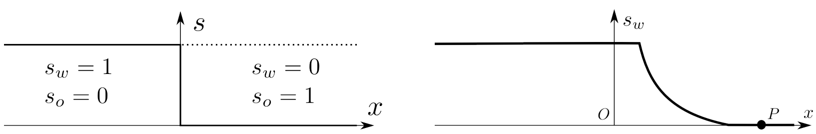
$$\phi \frac{\partial s_o}{\partial t} - \frac{\partial}{\partial x}\big(\frac{k s_o}{\mu_o}\frac{\partial p}{\partial x}\big) = 0, \tag{2.3b}$$

$$s_w + s_o = 1. \tag{2.3c}$$

Let $u = u_w + u_o$ denote the total flux.

Question 2b. *Show that u is constant in space.*

We consider the initial configuration described in Figure 2a where we have only water for negative x and only oil for positive x. We inject water from the left at a



(A) Initial Condition (B) The water penetrates in the oil region

Figure 2. One dimensional case.

constant rate, that is, we impose $\lim_{x\to-\infty} u(t,x) = \bar{u}$, for a constant $\bar{u} \geq 0$.

Question 2c. *Show that* $u(t,x) = \bar{u}$ *for all* t *and* x *and that the equations* (2.3a) *then reduce to*

$$\phi \frac{\partial s_w}{\partial t} + \frac{\partial}{\partial x}\big(f(s_w)\bar{u}\big) = 0 \tag{2.4}$$

and find the expression for f. *The function* f *is called fractional flow.*

For the initial data given above, equation (2.4) admits self similar solutions. In Figure 2b, we sketch the solution for a given time $t > 0$ and we observe that the water penetrates gradually in the oil region.

Question 2d (Open). *We consider a one dimensional reservoir which spans the half-line* $x \geq 0$. *At the beginning, it is full of oil, that is,* $s(0,x) = 0$ *for* $x \geq 0$. *We model the injection of water at* $x = 0$ *by setting* $s(0,x) = 1$ *for* $x < 0$. *We assume we have a production well located at a point* P *on the positive* x *axis (*$x_P > 0$*) and we start injecting water at* $x = 0$ *with a rate* $\bar{u}$. *Estimate the time it takes for the water to reach* P*, that is, the first time when* $s(t, x_P)$ *becomes non zero. Estimate the amount of water that goes through* P *before all the oil initially contained between* O *and* P *has been recovered. Relate your result to the viscosity ratio* μ_o/μ_w *so that you can comment on the difficulties highly viscous oil cause in oil recovery.*

For two phases flow, the Darcy law as given in (2.1) is in general not a good approximation and it is replaced by the following more general form

$$u_\alpha = -\frac{k_{r\alpha}(s_\alpha)}{\mu_\alpha} k \nabla p, \tag{2.5}$$

where the function $k_{r\alpha}$, which is called the *relative permeability*, is a given function of the saturation s_α, for each phase. Corey permeabilities, which are given by

$$k_{rw} = k_{rw}^0 s_w^{N_w} \quad \text{and} \quad k_{ro} = s_o^{N_o},$$

for some constants N_w, N_o, k_{rw}^0, are commonly used.

Question 2e (Open). *Set up and implement a numerical scheme to solve* (2.4). *Note that Equation* (2.4) *is hyperbolic and the numerical computation of its solution therefore requires an upwind method. Compute the solution for* (2.4) *with the same initial data as in Question 2d but now for nonlinear Corey permeabilities (take* $N_w = N_o = 2$ *and* $k_{rw}^0 = 1$*). Comment on the differences between the solutions for linear relative permeabilities (*$N_w = N_o = 1$ *and* $k_{rw}^0 = 1$*, as in Question 2d) and nonlinear relative permeabilities.*

3. The cost of heterogeneity

In a reservoir, the variation of the permeability of the rock is usually very important. A typical example being a fracture: The permeability within the fracture is very high compared to the neighboring rock. Such heterogeneity in the reservoir complicates recovery as the water which is injected travels fast through the region of high permeability and reach the producing wells possibly long before the oil which is trapped in the region of low permeability. This induces an extra cost as the oil needs to be separated from water in the producing well and more water has to be injected. Let us set up a simple model for which this cost can be estimated. We consider a two dimensional box consisting of horizontal layers with different permeabilities, see Figure 3. On the left and right sides, we have injecting and producing vertical wells with constant pressure equal to p_i on the left and p_o on the right ($p_i > p_o$). Let c_1 be the price of oil per $1\,\mathrm{m}^3$ and c_2 be the cost of producing $1\,\mathrm{m}^3$ of water (which corresponds to the cost for separation).

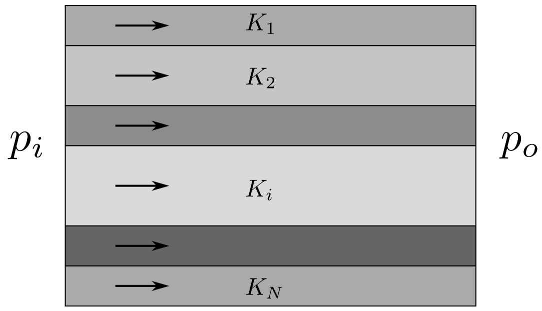


FIGURE 3. Reservoir consisting of horizontal layers with different permeability. We inject water from the left. Oil is produced from the right.

We consider the same model as in Section 2, still using a scalar permeability but now depending on x. We obtain the mass conservation equations

$$\frac{\partial s_\alpha \phi \rho_\alpha}{\partial t} - \nabla \cdot \big(\frac{\rho_\alpha s_\alpha}{\mu_\alpha} k(x,y) \nabla p \big) = 0, \tag{3.1}$$

for each phase $\alpha = \{w, o\}$. We assume that the fluids and the rock are incompressible and that the oil and water have the same viscosities, $\mu_w = \mu_o$. Then, the equations (3.1) reduce to the following equation for the pressure

$$\nabla \cdot (k(x,y) \nabla p) = 0. \tag{3.2}$$

Question 3a. *Derive the mass conservation equations* (3.1) *and the pressure equation* (3.2).

Question 3b (Open). *Use equation* (3.2) *to compute the oil production rate and the operation costs, as functions of time. Justify the fact that the equation decouple for each layer. Consider then the case when $N \to \infty$ and the width of each layer goes to zero, that is, the case where the permeability depends only on y, $k(x,y) = k(y)$.*

We now look at a multidimensional case, here a 2D square with an injecting well in the middle and four producing wells located at each corner, also called *five spots reservoir*, see Figure 4

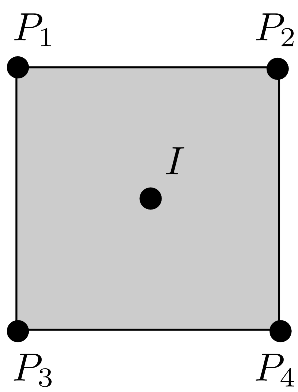


FIGURE 4. A five spots reservoir, with four production wells P_i, $i = 1, \ldots, 4$ at the corners and an injection well I in the middle.

Question 3c (Open). *Given a permeability distribution $k(x,y)$, derive and implement a finite volume formulation for this 2D reservoir. In particular, propose a* *way to compute the transmissibilities between adjacent cells when they have different permeabilities (Look first at the one dimensional case).*

The time of flight corresponds to the time for a particle in the flow to travel between two points (here from the injecting well to one of the producing wells). To solve (3.2), we use Matlab Reservoir Simulation Toolbox[1], an open access simulation code developed at Sintef for solving multiphase flow in porous media. We compute the solution to the pressure equation (3.2) and the time of flight. The code can be downloaded from the website of the course. The results are presented in Figure 5 and 6 for two types of reservoirs, one with constant permeability and the other with a randomly generated permeability. In Figure 7, we plot some streamlines that we numerate. In Tables 1 and 2, the time of flight for each streamline and the fluxes between the streamlines are given.

Question 3d (Open). *Give a mathematical definition of time of flight. Use the data in Tables 1 and 2 to estimate the operation costs. Compare the costs for the two reservoirs. Use the Matlab code to generate more streamlines. Set up other cases, with different permeability distribution, and compare the results.*

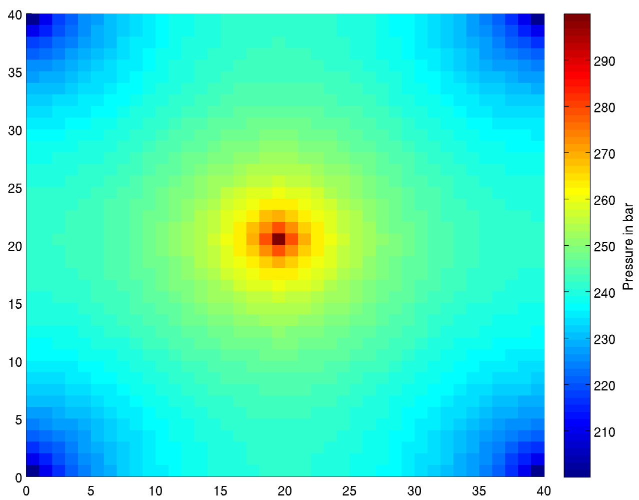
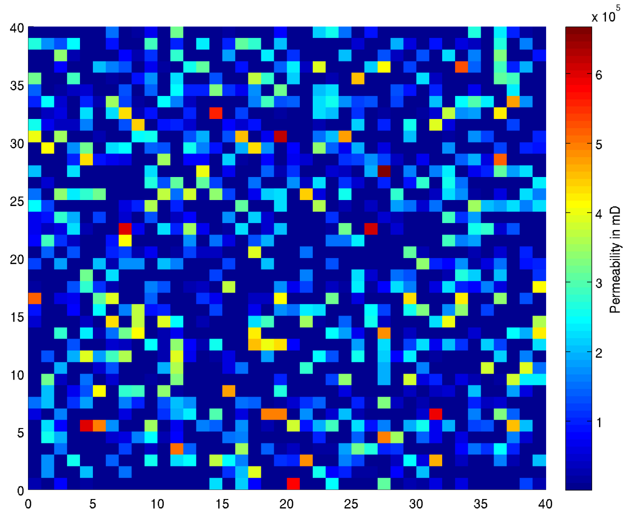
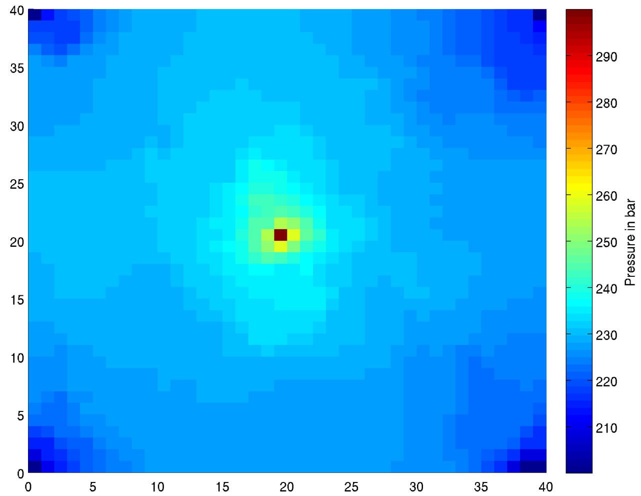


FIGURE 5. Pressure for case with uniform permeability

[1]http://www.sintef.no/Projectweb/Mrst/

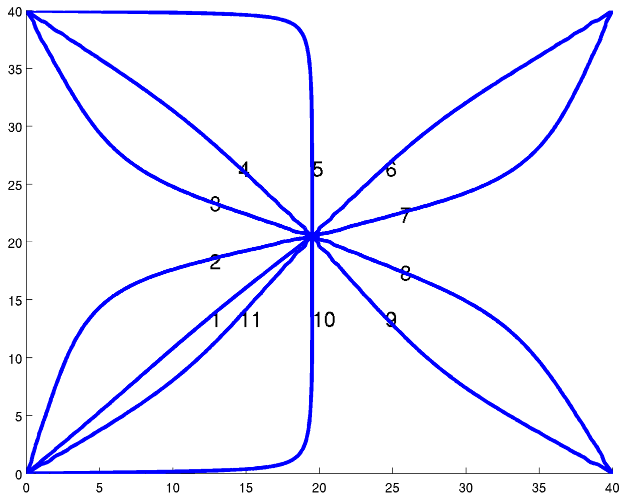


(A) Permeability

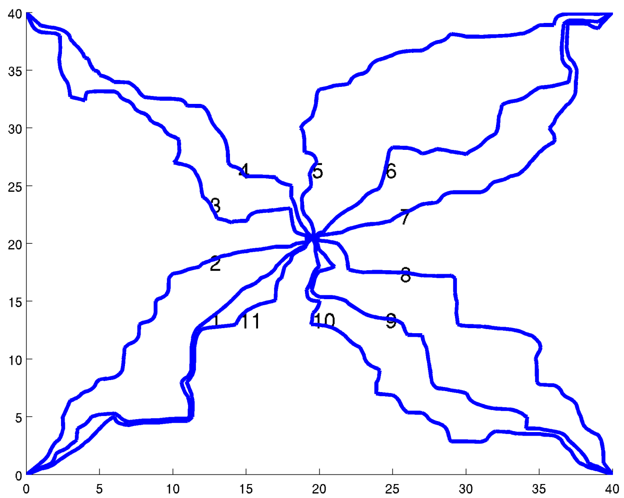


(B) Pressure

FIGURE 6. Case with randomly generated permeability



(A) Uniform case



(B) Random case

FIGURE 7. Streamlines

TABLE 1. Time of flight (in days) for given streamlines.

Streamline	Uniform case	Random case
1	1.5941015e+01	9.0438300e+02
2	2.1660807e+01	9.0523031e+02
3	1.7417845e+01	6.4953383e+02
4	1.5094670e+01	5.1394315e+02
5	4.6678338e+01	1.4114392e+03
6	1.6075629e+01	7.3539674e+02
7	1.9827070e+01	8.2575785e+02
8	1.8775226e+01	6.8812618e+02
9	1.7321176e+01	8.2367322e+02
10	4.9462737e+01	1.3325871e+03
11	1.6265244e+01	8.4871740e+02

TABLE 2. Flux in (in m^3/day) the streamline regions.

Streamline	Uniform case	Random case
1-2	9.4814770e+00	9.5715519e-02
2-3	9.0815331e+00	6.1852181e-01
3-4	9.7821966e+00	3.3314700e-01
4-5	1.1179679e+01	3.2576263e-01
5-6	1.0782385e+01	2.3522192e-01
6-7	1.0776050e+01	1.3414368e-01
7-8	1.0830640e+01	1.6854593e-01
8-9	1.0480024e+01	2.2661259e-01
9-10	9.7365972e+00	2.2514665e-01
10-11	9.3639352e+00	2.2613517e-01
11-1	3.7768601e+00	3.5882577e-02