Dimensional Analysis - Concepts

- **Physical quantities:** $R_j = v(R_j)[R_j] =$ value $\cdot$ unit, $\quad j = 1, \dots, m$.
- **Units:** $[R_j] = F_1^{a_{1j}} \cdots F_n^{a_{nj}}$, $\quad F_1, \dots, F_n$ fundamental units.
- **Dimension matrix** of $R_1, \cdots, R_m$: $A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$
- **Change of units** $\Rightarrow$ change of values:

  Lemma 1: $F_i = x_i \hat{F}_i$, $x_i > 0 \quad \Rightarrow \quad \hat{v}(R_j) = x_1^{a_{1j}} \dots x_n^{a_{nj}} v(R_j)$
- **Dimensionless combination:** $\pi = R_1^{\lambda_1} \cdots R_m^{\lambda_m}$ if $\vec{\lambda} \neq 0$, $[\pi] = 1$.
- **Dimensional independent** $R_1, \dots, R_s$ if no dimensionless comb'ns exists.
- **Physical relations** $\Phi(R_1, \dots, R_m) = 0$ are dimensionally consistent, i.e.

$$\Phi(v(R_1), \dots, v(R_m)) = 0 \quad \Leftrightarrow \quad \Phi(\hat{v}(R_1), \dots, \hat{v}(R_m)) = 0$$

for all changes of units $\hat{F}_i$. (consistent under change of units)

Dimensional Analysis - Buckingham's pi-theorem

(A1) $F_1, \ldots, F_n$ are fundamental units

(A2) $R_1, \ldots, R_m$ are physical quantities

(A3) $\Phi(R_1, \ldots, R_m) = 0$ is dimensionally consistent.

Lemma 2: Let $r = \operatorname{rank} A$, then $R_1, \ldots, R_m$ have $m - r$ independent dimensionless combinations.

OBS: The rank = number of linearly independent collumns in the matrix.

Buckingham's pi-theorem:

If (A1) – (A3) hold, then there are independent dimensionless combinations $\pi_1, \ldots, \pi_{m-r}$ and a relation Ψ such that

$$\Phi(R_1, \ldots, R_m) = 0 \quad \Leftrightarrow \quad \Psi(\pi_1, \ldots, \pi_{m-r}) = 0,$$

where $r = \operatorname{rank} A$ and A is the $n \times m$ dimension matrix of $R_1, \ldots, R_m$.

It remains to prove Lemma 2 and the Pi-theorem.