Equilibrium points

1. **Equilibrium point:**
   A constant solution u_e of the problem (e.g. ODEs or PDEs)

2. **Stable** equilibrium point u_e:
   All solutions $u(t)$ starting near u_e, remain near u_e for all $t \geq 0$:
   $\forall \varepsilon > 0 \; \exists \delta > 0 \quad \text{s.t.} \quad |u(0) - u_e| \leq \delta \Rightarrow |u(t) - u_e| < \varepsilon \quad \forall t > 0$

3. **Linear stability analysis** for u_e
   1. set solution $u = u_e + \tilde{u}$, $\quad |\tilde{u}| \ll 1$ small perturbation
   2. insert into equation, drop small(=non-linear) terms
      $\longrightarrow$ linear equation(s) for $\tilde{u}$ $\quad$ (= linearized equation(s))
   3. Check if all solutions of linearized equation(s) starting near 0 (= all small perturbations) remain near.
      $\longrightarrow$ If yes (no): indicate that u_e is stable (unstable).

4. Over time all physical systems tend to be at their stable equilibrium solutions! (... always small disturbances ...)

Aggregation of Amoeba

Background: *Lack of food → amoeba produce attractant and aggregate.*

Question:

Can onset of aggregation be caused by simple, uninteligent mechanism?

Model near onset of aggregation:

- Physical quantities:
  $a(x,t)$, $c(x,t)$ = amoeba, attractant densities; parameters
- Modelling (conservation+diffusion+production):

$$(1) \qquad a_t = \frac{\partial}{\partial x}\Big(ka_x - lac_x\Big), \qquad c_t = Dc_{xx} + q_1 a - q_2 c.$$

- Equilibrium points (=constant solutions):
  Constants (a_0, c_0) such that $q_1 a_0 = q_2 c_0$.
- Linearized equation around (a_0, c_0): (linear stability)
  $a = a_0 + \tilde{a}$, $c = c_0 + \tilde{c}$; $\tilde{a}, \tilde{c}$ small; drop small terms

$$(2) \qquad \tilde{a}_t = \frac{\partial}{\partial x}\Big(k\tilde{a}_x - la_0\tilde{c}_x\Big), \qquad \tilde{c}_t = D\tilde{c}_{xx} + q_1\tilde{a} - q_2\tilde{c}.$$

Aggregation of Amoeba

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- Particular solutions of (2): Fourier modes/eigenfunctions

$$(\tilde{a}, \tilde{c}) = e^{\alpha t}\cos(\beta x)(C_1, C_2)$$

solve (2) iff $\alpha^2 + b\alpha + c = 0$ for

$$b = k\beta^2 + D\beta^2 + q_2 \quad \text{and} \quad c = kq_2\beta^2 + kD\beta^4 - q_1 la_0\beta^2,$$

and then (C_1, C_2) satisfy two linear equations (last time).

Every $\beta \in \mathbb{R} \Rightarrow$ two real α, arbitrary small (C_1, C_2) and sol'ns $(\tilde{a}, \tilde{c})$

- Stability of solutions of (2):

$(\tilde{a}, \tilde{c})$ stable $\Leftrightarrow \alpha \le 0 \Leftrightarrow c \ge 0 \quad \Leftrightarrow \quad \boxed{kD\beta^2 + kq_2 \ge q_1 la_0}$

- Instability: $kq_2 < q_1 la_0$

$\Rightarrow$ $(\tilde{a}, \tilde{c})$ unbounded $(\alpha > 0)$ for $\beta \ll 1$ and $|C_1|, |C_2| \ll 1$

$\Rightarrow$ $(0, 0)$ unstable equilibrium point of (2)

$\Rightarrow$ (a_0, c_0) linearly unstable equilibrium point of (1)