SHORT INTRODUCTION TO THE FINITE VOLUME METHOD AND ITS IMPLEMENTATION IN MRST

1. Introduction to finite volume method

This is a *very short* introduction to finite volume methods. Finite volume methods are popular because of their flexibility and the fact that they are based on the same simple physical principle as the equations they aim to approximate. Let us consider three types of equations. The first type is a hyperbolic conservation law:

$$\frac{\partial u}{\partial t} + \nabla \cdot (F(u)) = 0, \tag{1}$$

where the flux $F(u)$ is a function of the unknown u. The second type is a diffusion equation:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u), \tag{2}$$

where the diffusion coefficient D depend on x. We can rewrite (2) in the same form as (1) but, in this case, the flux F depends linearly on the gradient,

$$F = -D \nabla u. \tag{3}$$

The third type is an elliptic equation which corresponds to the diffusion equation at equilibrium,

$$\nabla \cdot (D \nabla u) = 0. \tag{4}$$

Let us start by setting up a discretization for a flux problem where, given a source term, we try to compute the corresponding flux based on a conservation principle. We look at the equation

$$\nabla \cdot F = q. \tag{5}$$

After integrating (5) over an arbitrary domain Ω, we obtain

$$\int_\Omega q(x)\, dx - \int_{\partial\Omega} F \cdot n \, dS = 0, \tag{6}$$

which - in words - means that, for the amount of the quantity we are considering here (it can be many things: mass, heat, momentum ...), we have

$$\left\{ \begin{array}{c} \text{Local} \\ \text{production} \\ \text{in } \Omega \end{array} \right\} - \left\{ \begin{array}{c} \text{What} \\ \text{comes out} \end{array} \right\} + \left\{ \begin{array}{c} \text{What} \\ \text{comes in} \end{array} \right\} = 0 \tag{7}$$

Now, let us proceed with a discrete analog. We consider an unstructured mesh, that is, a mesh where the cells are polygons with an arbitrary number of faces. In Figure 1, we plot two neighboring cells and $n_{i,j}$ denotes the exterior (with respect to cell Ω_i) normal of the face between cells Ω_i and Ω_j. For the cell Ω_i, we introduce the average source term q_i, defined as

$$q_i = \int_{\Omega_i} q(x)\, dx.$$

Integrating (5), as we did to obtain (6), over the domain Ω_i, we obtain

$$\sum_{j \in N(i)} \int_{\partial\Omega_{i,j}} F \cdot n_{i,j} \, dS = q_i. \tag{8}$$

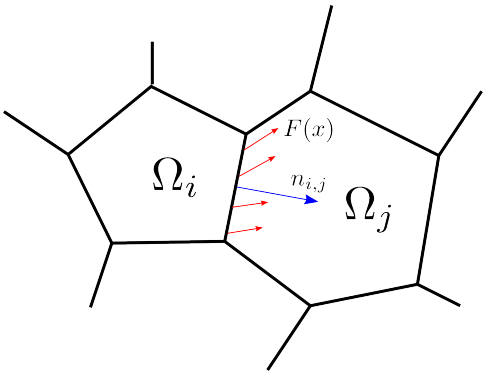

FIGURE 1. Two neighboring cells of an unstructured mesh.

Here, $N(i)$ denotes the index of the neighboring cells of Ω_i and $\partial\Omega_{i,j}$ denotes the intersection between Ω_i and Ω_j. Now, we have to define a way to approximate the integrals in (8). This will be done in different ways but we end up with an numerical flux $F_{i,j}$ on the face $\partial\Omega i, j$, which approximate the integrated flux on the face, namely,

$$F_{i,j} \approx \int_{\partial\Omega_{i,j}} F \cdot n_{i,j}\, dS. \tag{9}$$

After rewriting (8), we obtain

$$\sum_{j \in N(i)} F_{i,j} = q_i. \tag{10}$$

For (10), the summary given by (7) is as relevant as before. Let us apply the method to the elliptic equation (4). We decompose the equation in two,

$$\begin{cases} \nabla \cdot F = 0 \\ F = -D\nabla u. \end{cases} \tag{11}$$

The discretization of the first equation in (25) follows from (10) and we have

$$\sum_{j \in N(i)} F_{i,j} = 0. \tag{12}$$

We have to choose an approximation for the numerical fluxes $F_{i,j}$, as defined by (9), which in this case is

$$F_{i,j} \approx -\int_{\partial\Omega_{i,j}} D\nabla u \cdot n_{i,j}\, dS. \tag{13}$$

In reservoir simulation, it is common to use a two point flux approximation (TPFA), which is the simplest approximation one can consider. This approximation is given by a single interface coefficient depending on the permeability (which corresponds to D in the notation introduced here) and on the geometry. This coefficient is called the *transmissibility* and is defined such that

$$F_{i,j} = -T_{i,j}(u_j - u_i) \approx -\int_{\partial\Omega_{i,j}} D\nabla u \cdot n\, dS \tag{14}$$

Here, u_i and u_j are average values on cells, that is,

$$u_i = \frac{1}{V(\Omega_i)} \int_{\Omega_i} u\, dx, \tag{15}$$

where $V(\Omega_i)$ denotes the volume of the cell Ω_i. The numerical scheme defined by

$$\begin{cases} \sum_{j\in N(i)} F_{i,j} = 0 \\ F_{i,j} = -T_{i,j}(u_j - u_i), \end{cases} \tag{16}$$

which we can rewrite in the compact form

$$-\sum_{j\in N(i)} T_{i,j}(u_j - u_i) = 0 \tag{17}$$

is thus a finite volume approximation of the elliptic equation (4). To solve the diffusion equation (2), we have to discretize in time. We consider a time discretization parameter Δt and set $t^n = n\Delta t$. First we integrate (2) over a cell Ω_i and obtain

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\, dx = \sum_{j\in N(i)} F_{i,j}, \tag{18}$$

where $F_{i,j}$ denote the numerical fluxes as introduced earlier. Then, we make a first order approximation of the left-hand side in (18),

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\, dx \approx V(\Omega_i)\frac{u_i^{n+1} - u_i^n}{\Delta t}, \tag{19}$$

where

$$u_i^n = \int_{\Omega_i} u(t^n, x)\, dx. \tag{20}$$

For stability reason, which we do not develop here, the right-hand side in (18) has to be evaluated at time $n+1$ (we need an implicit scheme). Then using, the TPFA approximation, we get the following scheme

$$\begin{cases} u_i^{n+1} - u_i^n = \dfrac{\Delta t}{V(\Omega_i)} \sum_{j\in N(i)} F_{i,j}^{n+1}, \\ F_{i,j}^{n+1} = -T_{i,j}(u_i^{n+1} - u_j^n). \end{cases} \tag{21}$$

A finite different volume formulation for the conservation law (1) can be derived using the same principle. The difficulty in this case is that the solutions of (1) are typically discontinuous and the choice of the numerical flux becomes a highly nontrivial question.

2. Implementation in MRST

In this section, we describe briefly a finite volume discretization on unstructured grid for the Poisson equation

$$-\nabla\cdot(K(x)\nabla p) = q, \tag{22}$$

where q is a source term. The grid structure in MRST consists of three objects: The *cells*, the *faces* and the *nodes*. Each cell corresponds to a set of faces and each face to a set of *edges*, which are determined by the nodes. Each object has given geometrical properties (volume, areas, centroids). Let us denote by n_c and n_f, the number of cells and faces, respectively. There are two mappings, which determine the topology and which will be particularly used. The first one is given by $N : \{1, \dots, n_c\} \to \{0, 1\}^{n_f}$ and maps a cell to the set of faces which constitute this cell. The second one consists in fact of two mappings which, for a given face, give the corresponding neighboring cells, $N_1, N_2 : \{1, \dots, n_f\} \to \{1, \dots, n_c\}$. Let us now construct the discrete versions of the divergence and grad operators, which we denote `div` and `grad` . The mapping `div` is a linear mapping from faces to cells. We consider a discrete flux $u \in \mathbb{R}^{n_f}$. For a face f, the orientation of the flux $u[f]$ is

from $N_1(f)$ to $N_2(f)$. Hence, the total amount of matter leaving the cell c is given by

$$\texttt{div}(u)[c] = \sum_{f \in N(c)} 1_{\{c=N_1(f)\}} u[f] - \sum_{f \in N(c)} 1_{\{c=N_2(f)\}} u[f]. \tag{23}$$

The $\texttt{grad}$ mapping maps $\mathbb{R}^{n_c}$ to $\mathbb{R}^{n_f}$ and it is defined as

$$\texttt{grad}(p)[f] = p[N_2(f)] - p[N_1(f)], \tag{24}$$

for any $p \in \mathbb{R}^{n_c}$. Until now, the boundary conditions have been ignored. They are introduced by considering external faces and extending the mappings N_1 and N_2 to $S_c \cup \{0\}$ so that, if a given face f satisfies $N_1(f) = 0$ or $N_2(f) = 0$ then it is external. Note that the $\texttt{grad}$ operator only defines values on internal faces. We rewrite (22) in a mixed form

$$\nabla \cdot u = q \tag{25a}$$

$$u = -K(x)\nabla p. \tag{25b}$$

The vector $u \in \mathbb{R}^{n_f}$ is a discrete approximation of the flux on faces. Given $f \in S_f$, we have

$$u[f] \approx \int_{A_f} u(x) \cdot n_f \, ds,$$

where n_f is the normal to the face f, where the orientation is given by the grid. The relation between the discrete pressure $p \in \mathbb{R}^{n_c}$ and the discrete flux is given by a two point flux approximation,

$$u[f] = -T[f]\,\texttt{grad}(p)[f] \approx -\int_{A_f} K(x)\nabla p \cdot n_f \, dS, \tag{26}$$

where $T[f]$ denotes the *transmissibility* of the face f. Hence, the discretization of (25) is

$$\texttt{div}(u) = q \tag{27a}$$

$$u = -T\,\texttt{grad}(p). \tag{27b}$$

The multiplication in (27b) holds element-wise. Let us consider a two phase model and, for each phase $\alpha \in \{w, o\}$, the mass conservation equation

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t} - \nabla \cdot (\rho_\alpha \lambda_\alpha(s_\alpha) K(x) \nabla p_\alpha) = 0. \tag{28}$$

For the simplicity of the presentation, we assume here that the fluids are incompressible. We discretize the saturations as a vector $s_\alpha \in \mathbb{R}^{n_c}$. We have to discretize the flux $u_\alpha = -\lambda_\alpha(s_\alpha)K(x)\nabla p_\alpha$. To do so, we need to compute a face valued mobility vector $\lambda_\alpha^f \in \mathbb{R}^{n_f}$. Then, we define, for each face $f \in S_f$,

$$u_\alpha[f] = -\lambda_\alpha^f[f]\, T[f]\, \texttt{grad}(p)[f] \tag{29}$$

and an implicit scheme for (28) is given by

$$\frac{\phi}{\Delta t}(s_\alpha^{n+1} - s_\alpha^n) + \texttt{div}(u_\alpha^{n+1}) = 0. \tag{30}$$

To determine the face valued mobility, we use an upwind approximation, that is, we set

$$\lambda_\alpha^f[f] = \begin{cases} \lambda_\alpha(s_\alpha[N_1(f)]) & \text{if } u_\alpha[f] > 0, \\ \lambda_\alpha(s_\alpha[N_2(f)]) & \text{otherwise .} \end{cases} \tag{31}$$