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Exercise set 1
Exercise session 2015-8-26

$\boxed{1}$ In this exercise we are going to prove Buckingham's Pi theorem. Assume that $F_1, \ldots, F_l$ are fundamental units, $R_1, \ldots, R_n$ are physical quantites, and that A is the dimension matrix corresponding to the fundamental units and physical quantites. Let $r = \operatorname{rank}(A)$, and assume that there is a physically meaningful relation

$$\Phi(R_1, \ldots, R_n) = 0. \tag{1}$$

You will prove the theorem by proving each of the following lemmas. Please write the complete proof and hand it in to Anders. We will not grade this, but we will use your proofs to understand how far you have been able to reach on your own and how good your skills in linear algebra are. A complete proof following these steps will be published on the webpage.

Lemma 1

There are precisely $n - r$ independent dimensionless combinations $\pi_1, \ldots, \pi_{n-r}$.

Lemma 2

a) A change of units will not change the dimensionless combinations. That is given new units $\hat{F}_i = x_i F_i$ and any dimensionless combination π we have that $v(\pi) = \hat{v}(\pi)$ where v is the numerical value in units F_j and $\hat{v}$ is the numerical value in units $\hat{F}_j$.

b) If $v(R_1) > 0, \ldots, v(R_n) > 0$ there exists x_i's and new units F_i such that $v(R_1) = \ldots = v(R_n) = 1$.

Lemma 3

If $\pi_1, \ldots, \pi_{n-r}$ are $n-r$ independet dimensionless combinations of $R_1, \ldots, R_n$, then for each Φ satisfying (1) there exists a relation Ψ such that

$$\Phi(R_1, \ldots, R_n) = 0 \equiv \Psi(\pi_1, \ldots, \pi_{n-r}) = 0. \tag{2}$$

Hint proof of Lemma 1

We have $1 = [\pi_j] = R_1^{\lambda_1} \ldots R_n^{\lambda_n}$, and $[R_j] = F_1^{a_{1j}} \ldots F_l^{a_{lj}}$. This gives a system of equations. Take logarithms on both sides and we get a homogenous linear system $A\lambda = 0$. You want to find the dimension of the null space of A (why?).

Hint proof of Lemma 2

a) Use the formula for the change of value of R_j. Use the definition of dimensionless combination and find $\hat{v}(\pi)$. Show that $v(\pi) = \hat{v}(\pi)$.

b) Consider $y_j = \log(v(R_j)), b_j = \log(\hat{v}(R_j)$ and the equations for $\hat{v}(R_j)$. Take logarithms for both sides of these equations. When is this system solvable? How many solutions are there? Consider the case $b_j = 0$.

Hint proof of Lemma 3

Step 1 Since $\mathrm{rank}(A) = r$ there are r independent physical quantities among $R_1, \ldots, R_n$. We can assume that the independent quantities are $R_1, \ldots, R_r$. Prove that for $j = 1, \ldots, n-r$ there exist $\lambda_{1j}, \ldots, \lambda_{rj}$ such that

$$\frac{R_{r+j}}{R_1^{\lambda_{1j}} \ldots R_r^{\lambda_{rj}}} = \tilde{\pi}_{r+j}, \tag{3}$$

where $\pi_{\tilde{r+j}}$ is dimensionless. Use that $\pi_1, \ldots, \pi_{n-r}$ are independent dimensionless combinations to conclude that there are $\alpha_{1j}, \ldots, \alpha_{(n-r)j}$ such that

$$\tilde{\pi}_{r+j} = \pi_1^{\alpha_{1j}} \ldots \pi_{n-r}^{\alpha_{(n-r)j}}. \tag{4}$$

Step 2 From step one we know that there is a one-to-one correspondance $(R_1, \ldots, R_n) \to (R_1, \ldots, R_r, \pi_1, \ldots, \pi_{n-r})$. Define $\tilde{\Psi}$ by

(5)

$$\tilde{\Psi}(R_1, \ldots, R_r, \pi_1, \ldots, \pi_{n-r}) = \Phi(R_1, \ldots, R_r, \pi_1^{\alpha_{11}} \ldots \pi_{n-r}^{\alpha_{(n-r)1}} R_1^{\lambda_{11}} \ldots R_r^{\lambda_{r1}}, \ldots).$$

From Lemma 2 b) we know that there is a particular change of units such that $v(R_1) = \ldots = v(R_r) = 1$, and thus we can define

$$\Psi(\pi_1, \ldots, \pi_{n-r}) = \tilde{\Psi}(1, \ldots, 1, \pi_1, \ldots, \pi_{n-r}). \tag{6}$$

$\boxed{2}$ (Problem 4.1.8 p. 52 in Krogstad)
By measuring the pressure drop P in the feed pipe against the time t taken to fill a vessel of volume V, Bose, Bose and Ruert (around 1910) plotted the relations shown in Figure 1 for water, chloroform, boroform, and mercury. Show by dimensional analysis (using the density ρ and the dynamic viscosity [1] μ) that there should be one common relation that turns these curves into one. That is, find the variables along the axes of von Kármán's representation of the same data, as shown on the right hand figure.

[1]Kinematic viscosity is the ratio between dynamic viscosity μ and fluid density ρ, $\nu = \frac{\mu}{\rho}$. Dynamic viscosity μ has dimensions which can be derived from its definition: A *stress* is a force per area (of which pressure is an example), and shear stresses in a fluid are proportional to the velocity gradient (units: velocity per length). The dynamic viscosity μ is the constant of proportionality.

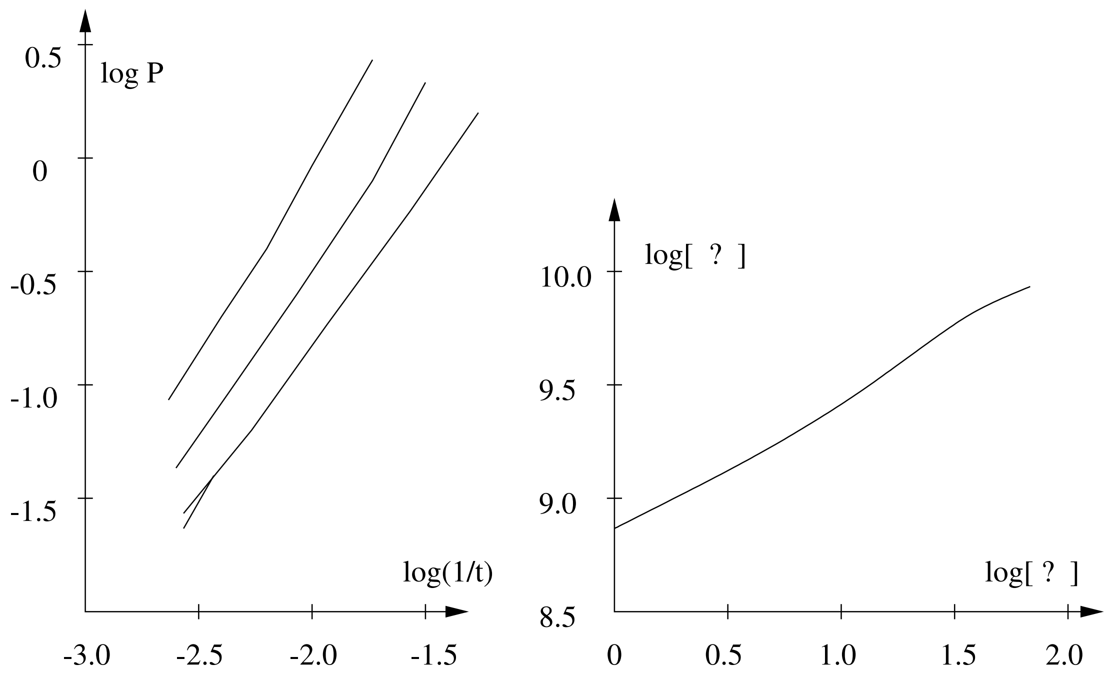

Figure 1: Presentation of the data in the original paper (left) and von Karmanś revised graph after applying dimensional analysis (right)