Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4195 Mathematical modelling Autumn 2015

Exercise set 2
Exercise session 2015-9-2

1 (Exersice 7 p. 204 in Lin and Segel)
For a small sphere falling under gravity in a viscous fluid, it is observed that the speed of fall is (after a short time) a constant v. Let a, ρ_1, ρ_2, μ, and g represent the radius and density of the sphere, the density and viscosity of the liquid, and the gravitational acceleration, respectively.

(a) Using dimensional analysis with fundamental units of mass, length, and time, show that

$$v = \frac{\mu}{\rho_1 a}\phi\left(\rho_1^2\mu^{-2}a^3 g, \frac{\rho_2}{\rho_1}\right) \tag{1}$$

Use the fact that viscous stress (force/unit area) equals the product of μ and the velocity gradient (derivative of velocity with respect to length).

(b) Since motion here is unaccelerated, we need not make use of the proportionality of acceleration to force, and force can be treated as a separate fundamental unit. Hence show that

$$v = a^2\rho_1 g\mu^{-1}\phi\left(\frac{\rho_2}{\rho_1}\right). \tag{2}$$

Stokes derived the formula

$$\phi(x) = \frac{2}{9}(1-x) \tag{3}$$

a result of great utility. For example, it was used in the Millikan oil drop experiment.[1]

2 (Exercise 8 page 205 from Lin and Segel)
Consider steady non turbulent incompressible flow of a liquid in a circular pipe. The pressure difference Δp between the two ends of the pipe should depend only on its length L, radius R, and the maximum speed U of the fluid of viscosity μ and density ρ. Show that according to dimensional analysis the situation can be described equally well by either of the following equations:

$$\frac{\Delta p}{\frac{1}{2}\rho U^2} = \phi\left(\frac{\rho U R}{\mu}, \frac{\rho U L}{\mu}\right),$$
$$\frac{\Delta p}{\frac{1}{2}\rho U^2} = \phi\left(\frac{\rho U R}{\mu}, \frac{L}{R}\right).$$

[1] Millikan used oil drops to determine the elementary electric charge. The idea is to put small drops of electrically charged oil between two horizontal charged plates and measure the electric force acting on the oil drops.

It is reasonable to suppose that when $\frac{L}{R}$ is large, say $\frac{L}{R} > 20$, changing $\frac{L}{R}$ should have little effect on the answer. In this case the relation

$$\frac{\Delta p}{\frac{1}{2}\rho U^2} = \phi\left(\frac{\rho U R}{\mu}\right). \tag{4}$$

should hold to good accuracy. This illustrates the fact that physical reasoning sometimes shows how to choose the dimensionless parameters so that one of them can be neglected. For more examples of this type of reasoning, see the valuable book of Kline (1965), from which the present problem was adapted. Use $[\mu] = \frac{\mathrm{kg}}{\mathrm{m\,s}}$.

3 (Problem 4.2.3 p. 54 in Krogstad)
A common mathematical model for the size of a population $y^*(t^*)$ as a function of time t^* is described by the logistic equation

$$\frac{\mathrm{d}y^*}{\mathrm{d}t^*} = ry^*\left(1 - \frac{y^*}{K}\right).$$

Here r is called the growth rate and K the sustainable capacity.

(a) Which scale is suitable for y^*?

(b) Determine a time scale when $y^* \ll K$.

(c) Introduce these scales into the equation so that it becomes dimensionless (The equation can easily be solved by inserting $y = \frac{1}{u}$ and solving for u).

4 A spherical bullet is dropped in a viscous Newtonian fluid in a container with depth L. The bullet starts at a position x_0^* with initial velocity v_0^*, and at time t^* is at position $x^*(t^*)$. The bullet has mass M and radius r, the density of the fluid is ρ, and the kinematic viscosity ν.

(a) Show by dimensional analysis that when the bullet has constant velocity v^* the friction force F_D acting on the bullet is of the form

$$F_D = \nu\rho r v^* \phi\left(\frac{v^* r}{\nu}\right). \tag{5}$$

(b) Assume that the function ϕ in (5) is given by $\phi(x) = \bar{k}$ for some constant $\bar{k}$ and that (5) holds when the bullet accelerates, and use Newton's second law to model the fall of the bullet. [2]

(c) Scale the equation you get in (b) in the case when friction is strong and the bullet has constant velocity for most of the fall. Find appropriate values for the scales X and T. Find the biggest coefficient and divide the equation by it. Scale the initial conditions and find μ such that $\dot{x}(0) = \mu$. You will find that one of the terms in the equation is multiplied by a small parameter ε. Find ε expressed by v_l and v_f.[3]

[2] The forces acting on the bullet are gravity $F_G = Mg$, friction, and buoyancy $F_B = -M\frac{\rho_{fluid}}{\rho_{bullet}}g$.

[3] We have $v_l := \frac{M\tilde{g}}{k}$ is the equilibrium velocity, $v_f := \sqrt{L\tilde{g}}$ is the free fall velocity, and v_0 is the unscaled initial velocity.

(d) Simplify the equation by setting $\varepsilon = 0$. Solve and call the solution $x_0(t)$, and find the corresponding unscaled solution $x_0^*(t^*)$. What happens to the initial conditions? Verify by differentiation that the exact solution of the bullet problem is

$$x^*(t^*) = \frac{M\tilde{g}}{k}t^* + \frac{M}{k}\left(\frac{M\tilde{g}}{k} - v_0\right)\left(e^{-\frac{k}{M}t^*} - 1\right) + x^*(0). \tag{6}$$

What can you conclude after comparing $x_0^*(t^*)$ with the exact solution $x^*(t^*)$?

(e) Sometimes the approximation $\phi(x) = \bar{k}$ is not good enough. Assume that the function ϕ in (5) is given by the expression $\phi(x) = k_1 + k_2|x|$, where the absolute value comes from the fact that the friction force should reduce the speed. Assume that $\frac{\mathrm{d}x^*}{\mathrm{d}t^*}(0) > 0$, and use Newton's second law to model the fall of the bullet, and scale the equation assuming that

$$M\frac{X}{T^2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \sim \nu\rho r k_1 \frac{X}{T}\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{7}$$

Show that if we use the scale $X = \frac{M^2\tilde{g}}{(\nu\rho r k_1)^2}$ we get an equation of the form

$$\ddot{x} = 1 - \dot{x} - \varepsilon(\dot{x})^2, x(0) = 0, \qquad \dot{x}(0) = u. \tag{8}$$

Assume that $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$, insert into (8) and find $x_0(t)$ and $x_1(t)$.