TMA4195
Mathematical modelling
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Exercise set 4
To be handed in 2015-10-7

1 (a) The number of individuals in a population N^* is often modelled in terms of a logistic model,

$$(1) \qquad \frac{1}{N^*(t^*)}\frac{dN^*(t^*)}{dt^*} = r\left(1 - \frac{N^*(t^*)}{N_m}\right), \quad 0 < r, \quad 0 < N_m.$$

Scale the equation and show how to investigate the stability of the equilibrium points.

In closed laboratory studies of bacteria it has turned out to be difficult to find populations following (1). Instead, the population, after a while, tends to fall to 0 due to self-poisoning. This probably also applies to mankind. The contamination may be due to PCBs, long-living radioactive waste and hormones influencing fertility. The following alternative model is therefore proposed for the population of the Earth:

$$(2) \qquad \frac{1}{N^*(t^*)}\frac{dN^*(t^*)}{dt^*} = r\left(1 - \frac{N^*(t^*)}{N_m}\right) - c\int_{-\infty}^{t^*} N^*(s^*)\,ds^*, \quad c > 0.$$

(b) How can we argue for the last term? Show that for this model, the only possible limit when t^* tends to infinity is $N^* = 0$.

(c) Assume that N_m is so large that N^* never approaches N_m, scale the equations and show that we may approximately write

$$(3) \qquad \frac{1}{N}\frac{dN}{dt} = 1 - \alpha\int_{-\infty}^{t} N(s)\,ds.$$

Solve (3) by introducing $P(t) = \alpha\int_{-\infty}^{t} N(s)\,ds$, and determine how the population and the pollution develop over time.

Hint: The equation $y'' = y'(1-y)$ has, when $\lim_{x\to-\infty} y(x) = 0$, the general solution

$$y(x) = \frac{2}{1 + \mathrm{e}^{-(x-x_0)}} = 1 + \tanh\frac{x - x_0}{2},$$

and

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} = \frac{4}{(\mathrm{e}^x + \mathrm{e}^{-x})^2}.$$

2 In this exercise we will investigate the stability of satellites in orbit.

a) Let a satellite of mass m orbit the Earth, mass M. Assume that the only force acting on the satellite is gravity. Use Newton's second law to model the flight of the satellite in polar coordinates. [1] Scale the equations so we get

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{1}{r^2}, \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0. \end{aligned}$$

Which scales are used?

We observe that only the derivative of θ appears in the system, so we define $v = \dot{r}$ and $\omega = \dot{\theta}$, and so we get the system of equations

$$\dot{r} = v, \tag{4}$$

$$\dot{v} = r\omega^2 - \frac{1}{r^2}, \tag{5}$$

$$\dot{\omega} = -2\frac{\omega v}{r}. \tag{6}$$

b) Find the equilibrium points of (4). We want to investigate the stability of equilibria points with regular perturbation. To that end select an equilibrium (r_0, v_0, ω_0) and let

$$\begin{aligned} r(t) &= r_0 + \epsilon r_1(t) + O(\epsilon^2), \\ v(t) &= v_0 + \epsilon v_1(t) + O(\epsilon^2), \\ \omega(t) &= r_0 + \epsilon \omega_1(t) + O(\epsilon^2), \end{aligned}$$

with initial data $r_1(0) = 0$, $v_1(0) = \bar{v}$, and $\omega_1(0) = 0$. Solve for $r_1(t)$. What can we say about the stability of orbits?

3 (Exercise 2.2 p. 377 in Logan) Determine the equilibrium solutions and sketch a bifurcation diagram of the following differential equations. Identify the bifurcation points and bifurcation solutions. Investigate the stability of the equilibrium solutions and indicate where an exchange of stability occurs.

(a)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (u - \mu)(u^2 - \mu)$$

(b)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u(9 - \mu u)(\mu + 2u - u^2)$$

4 (Exercise 5 p. 299 in Lin & Segel)

Use singular perturbation theory to obtain outer, inner, and composite expansions to the solution of the problem

$$\epsilon u'' - (2 - x^2)u = -1, \qquad u(-1) = u(1) = 0.$$

REMARK. It is sufficient to solve the differential equation on $(0, 1)$ subject to the boundary conditions $u'(0) = 0, u(1) = 0$. Why?

[1] The gravitational force F_G on a mass m from a mass M is given by $F_G = -\frac{GMm}{r^2} e_r$, where r is the distance between the masses, G is the gravitational constant, and e_r is the unit vector in the direction from M to m.