Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4195
Mathematical modelling
Autumn 2015

Exercise set 5
To be handed in 2015-10-28

$\boxed{1}$ Use the method of characteristics to solve these initial value problems for $t > 0$.

(a) $u_t + u_x = -u, \qquad u(x, 0) = u_0(x)$
(b) $u_t + u_x = x, \qquad u(x, 0) = u_0(x)$

$\boxed{2}$ (Exercise 7.2.3 p 74 in Krogstad)

(a) The most common model for the traffic of cars along a road leads to a dimensionless flux of cars of the form $j = \rho(1-\rho)$. Describe how this model is set up. State the hyperbolic equation the model leads to (where no cars enter or leave the road). When will the solution develop shocks?

Consider the model in (a). Between $x = 0$ and $x = 1$ there is a reduction in the speed limit such that the maximum speed reduces to $1/2$, while the maximum density remains the same. We assume that the same type of relation between the car velocity and the density also applies for this part.

(b) Which condition on the flux of cars has to hold in $x = 0$ and $x = 1$? Find the solution $\rho(x, t)$ for $t > 0$ and all x when

$$\rho(x, 0) = \begin{cases} 1/2, & x < 1, \\ 0, & x > 1. \end{cases}$$

Hint: The density ρ is constant between 0 and 1 for all $t \geq 0$.

$\boxed{3}$ Exercise 7.2.1 in Krogstad
Many years ago, NTNUI (then NTHI) arranged the Student Mile (=10km) at the old Trondhjem stadion. The arrangement was terribly crowded with about 1000 participants, all starting (or trying to) at the same time. This spectacular event, often in rain and on a dirty track made of coke sand, *koksgrus*, is the origin of the following problem.

Consider a round-track with length $L = 400$m. We assume that the mean running speed v^* decreases linearly with the density ρ^*, so that $v^* = v^*_{\max}$ for $\rho^* = 0$ students/m, and $v^* = 0$ m/s when $\rho^* = \rho^*_{\max}$. We also consider the track to be one-dimensional (but round!).

(a) State the conservation law for students (no late entries or drop-outs!) under these conditions, introduce dimensionless variables, and show that the differential formulation may be written as

$$\rho_t + (1 - 2\rho)\,\rho_x = 0 \qquad (1)$$

where $0 \leq \rho \leq 1$ and $\rho(x,t)$ is a 2π-periodic function of x.

(b) Find (in implicit form) the exact solution of (1) when

$$\rho(x,0) = \rho_0 + \epsilon\cos(x), \quad (0 < \rho_0 \pm \epsilon,\ \epsilon > 0).$$

(c) Sketch the characteristics in the xt-plane for the solution in (b) and show that, as a solution of the integral conservation law, it breaks down and forms a shock in the density after a certain time.

(d) When will the shock start and what happens to the shock when $t \to \infty$?
(*Question (d) is difficult. Start by determining the crossing points for characteristics starting at* $3\pi/2 - \theta$ *and* $3\pi/2 + \theta$, *when* θ *varies from* 0 *to* π. *Try to prove that the crossing points lie on a straight line segment, and that there are no crossings elsewhere. Finally, check that the line segment is a shock and that the corresponding solution indeed satisfies the integral conservation law.*)