Laplace solver for irregular grids

Laplace equation

- The Laplace equation:

$$-\Delta p = h$$

with Neumann boundary conditions

$$\nabla p \cdot \boldsymbol{n} = 0.$$

- The solution to Laplace solves the minimization problem

$$\min \left(\frac{1}{2} \int_{\Omega} |\nabla p|^2 \, dx - \int_{\Omega} h(x) p(x) \, dx \right).$$

- Darcy's law:

$$\boldsymbol{v} = -\nabla p \quad (\boldsymbol{v} = -K \nabla p)$$

so that $\int_{\Omega} |\boldsymbol{v}|^2 \, dx$ corresponds to the **dissipative** energy.

Calculus of variation

- Let
$$F(p) = \frac{1}{2}\int_\Omega |\nabla p|^2 \, dx - \int_\Omega h(x)p(x)\,dx.$$
- Let p be the minimizer. Introduce a perturbation δp of the form $\delta p = \varepsilon \tilde{p}$
- We have:
$$F(p+\delta p) = F(p) + \varepsilon\left(\int_\Omega \nabla p \cdot \nabla \tilde{p}\,dx - \int_\Omega h\tilde{p}\,dx\right) + o(\varepsilon^2)$$

  Hence,
$$\frac{dF}{d\varepsilon}_{|\varepsilon=0} = \int_\Omega \nabla p \cdot \nabla \tilde{p}\,dx - \int_\Omega h\tilde{p}\,dx$$
- Since p is a minimizer,
$$\int_\Omega \nabla p \cdot \nabla \tilde{p}\,dx - \int_\Omega h\tilde{p}\,dx$$

Calculus of variation (ctd)

- After integration by parts, and using Neumann bc,

$$\int_\Omega (-\Delta p - h)\tilde{p}\, dx = 0,$$

which holds for **all** $\tilde{p}$. Hence, the Laplace equation.

- Let us now consider the minimization problem

$$\min \frac{1}{2} \int_\Omega |\boldsymbol{v}|^2 \, dx$$

with the **incompressibility** constraint

$$\nabla \cdot \boldsymbol{v} = h$$

- Introduce the Lagrange multiplier p and consider the augmented Lagrangian function

$$\mathcal{L} = \frac{1}{2} \int_\Omega |\boldsymbol{v}|^2 - \int_\Omega p \nabla \cdot \boldsymbol{v}\, dx.$$

Calculus of variation (ctd)

- Necessary conditions for a minimizer are

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{v}} = 0 \quad \text{and} \quad \frac{\delta \mathcal{L}}{\delta p} = 0$$

- Those condition imply

$$\boldsymbol{v} + \nabla p = 0 \quad \text{and} \quad \nabla \cdot \boldsymbol{v} = h,$$

  that is the Laplace equations.
- We did not require the Darcy's law!
- All the *physics* is in the formulation of the energy and the divergence operator.
- To obtain the governing equations, we simply use calculus of variation.

Finite volume methods

- Finite volume methods are based on flux approximations

$$v_\sigma \approx \int_\sigma \boldsymbol{v} \cdot \boldsymbol{n} \, dx.$$

- The discrete divergence operator is naturally given by

$$\mathtt{div}_K v = \sum_{\sigma \in \mathcal{F}_K} v_\sigma .$$

- It remains to approximate the energy

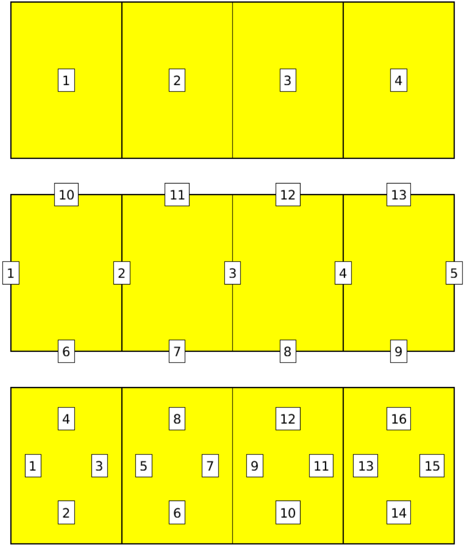
$$\frac{1}{2} \int_K |\boldsymbol{v}|^2 \, dx \approx \frac{1}{2} \sum_{\sigma_i, \sigma_j} v_{\sigma_i} B_{i,j} v_{\sigma_j}$$

- Then, we get the discrete equation by a variational approach.
- To get convergence of the methods, we need consistency and stability for B.

Unstructured grid

- Topological description
  - **cells** made of
  - **faces** made of
  - (**edges** made of)
  - **nodes**.
- Commands to access topology
  - `G.cells.faces`
  - `G.cells.facePos`
  - `G.faces.nodes`
- Within each topological categories, the elements are numbered.
- How are these entities numbered? We do not care!
- How do we find the number of a cell?

The elements are numbered



The discrete system

$$\begin{bmatrix} B & C & D & N \\ C^T & 0 & 0 & 0 \\ D^T & 0 & 0 & 0 \\ N^T & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ \pi \\ p_{\mathsf{neum}} \end{bmatrix} = \begin{bmatrix} -D_{\mathsf{dir}} p_{\mathsf{dir}} \\ h \\ 0 \\ 0 \end{bmatrix}$$

Flux on half faces : $v_{K,f} \approx \int_f \nabla p(x) \cdot \boldsymbol{n}_{K,f} \, dx$

Pressure in cell : $p_K \approx \frac{1}{|K|} \int_K p(x) \, dx$

Pressure on internal faces : $\pi_f \approx \frac{1}{|f|} \int_f p(x) \, dx$

Pressure on Neumann faces : $p_{\mathsf{neum},f} \approx \frac{1}{|f|} \int_f p(x) \, dx$

Pressure on Dirichlet faces : $p_{\mathsf{dir},f} \approx \frac{1}{|f|} \int_f p(x) \, dx$

The variables