Mathematical Modeling
Project fall 2015

Loen 1905, 1936

Loen, Ramnefjellet in background

- Two accidents: 1905, 1936
- one million cubic meter block
- 800 meter high
- 70 meter high wave

Åkneset

Åkneset (Storfjorden)

- Extensive monitoring
- approx. 54 million cubic meter
- 100m-900m

Tsunami wave in fjords

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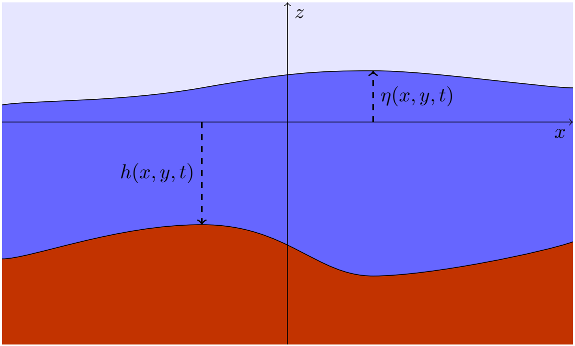
- Three types of work
  - Modeling (mod) : Derive the equations. Identify the determining parameters. Simplify the equations.
  - Analytic (ana) : Solve analytically simple equation.
  - Numerical (num) : Solve numerically more realistic models.

Project plan

1. The model equations
2. Reduction to a linear model
3. Generation of the tsunami
4. Inundation

The geometry



Conservation laws

- Conservation of mass and momentum for mass point

$$\frac{dm}{dt} = 0, \quad \frac{d}{dt}(m\boldsymbol{v}) = 0,$$

more precisely

$$m\frac{d\boldsymbol{v}}{dt} = \mathbf{f}.$$

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- For a fluid (continuum) model, it becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0,$$
$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p - \rho \boldsymbol{g}.$$

The forces are pressure and gravity.

Irrotational flow

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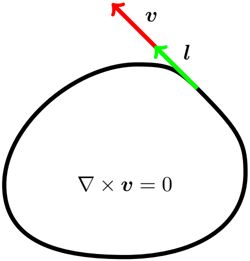
- Proof: assume the opposite. Then along a closed curve γ, $\boldsymbol{v} = |\boldsymbol{v}|\,\boldsymbol{t}$ with $|v| > 0$.

We have
$$\int_\gamma \boldsymbol{v} \cdot d\boldsymbol{l} = \int_\gamma |\boldsymbol{v}|\,|\boldsymbol{t}|^2\,dl = \int_\gamma |\boldsymbol{v}|\,dl > 0$$
and
$$\int_\gamma \boldsymbol{v} \cdot d\boldsymbol{l} = \int_\Gamma (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{n} ds = 0$$



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because $\gamma(0) = \gamma(1)$
- For an incompressible fluid, mass conservation gives us
$$\nabla \cdot \boldsymbol{v} = 0.$$
Hence, ϕ satisfies the Laplace equation
$$\Delta\phi = 0.$$

Governing equation for irrotational flow

- The governing equations reduce to

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$$\nabla\phi \cdot \boldsymbol{n} = 0$$

at the bottom,

$$0 = \eta_t + \nabla\phi \cdot [\eta_x, \eta_y, -1] \quad \text{and} \quad \frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + g\eta = 0$$

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- Note that the conservation of momentum equation is *integrated*. The equation for pressure decouples.

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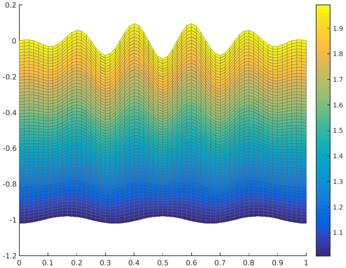

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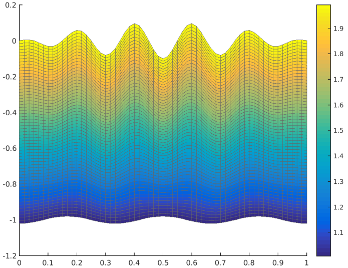

- Propose a numerical scheme for the potential equations.

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on an interval $[0, L]$.

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- General solutions using superposition principle

$$u(t,x) = \sum_{n\geq 0} a_n u_n(t,x).$$

dispersion relation

- Plane wave

$$u(t,x) = e^{2\pi i(kx-\omega t)}$$

where

$$\begin{aligned} k &: \text{ wavelength,} \\ \omega &: \text{ frequency.} \end{aligned}$$

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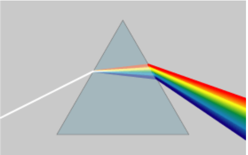
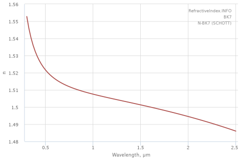
- Refraction





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- How much energy?
- From this value, how to set up the initial condition for the wave equation?

Shallow water equation

- Assuming that

  $$\text{depth} \ll \text{typical wavelength}$$

  the potential equations are approximated at first order by the **shallow water** equations,

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- System of conservation laws
- There exist $V(u,\eta)$, $W(u,\eta)$ (Riemann invariants) such that the equations become *less* coupled.

$$\begin{pmatrix} u \\ \eta \end{pmatrix}_t + \begin{pmatrix} F(u,\eta) \\ G(u,\eta) \end{pmatrix}_x = 0 \iff \begin{cases} V_t + \lambda_1 V_x = 0 \\ W_t + \lambda_2 W_x = 0 \end{cases}$$

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- Example on GPU.

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- Keep the balance between

  modeling - *analytic* - *numerics*.