MIXED FORMULATION AND MIMETIC FINITE DIFFERENCE

The goal of this note is to introduce the mimetic finite difference method, which is a convergent numerical method which can handle general grids (see [1] for a complete reference). We will use it to solve the Poisson equation

$$-\Delta p = h, \tag{1}$$

for some given source term h. We will follow a variational approach which brings also some interesting insight into the equation.

1. Variational approaches for the Poisson equation

For the moment, we will assume Neumann boundary conditions, that is, $\nabla p \cdot \mathbf{n} = 0$ on the boundary $\partial\Omega$ of the domain Ω.

We consider the minimization problem

$$\min \left(\frac{1}{2} \int_\Omega |\nabla p|^2 \, dx + \int_\Omega h(x) p(x) \, dx \right). \tag{2}$$

A necessary condition for $p(x)$ to be solution to (2) can be obtained using a variational approach. Let $F(p) = \frac{1}{2} \int_\Omega |\nabla p|^2 \, dx + \int_\Omega h(x)p(x)\, dx$. For *any* perturbation function $\delta p(x)$, the function $\varepsilon \mapsto F(p + \varepsilon \delta p)$ admits a minimum at $\varepsilon = 0$. Hence, we have

$$\frac{dF}{d\varepsilon}_{|\varepsilon=0} = \int_\Omega \nabla p \cdot \nabla \delta p \, dx + \int_\Omega h(x) \delta p(x) \, dx = 0. \tag{3}$$

The perturbed function $p + \varepsilon \delta p$ has to satisfy the boundary condition. It implies that $\nabla \delta p \cdot \mathbf{n} = 0$ at the boundary of the domain. After integration by part, we obtain that

$$\int_\Omega (-\Delta p + h(x)) \delta p(x) \, dx = 0$$

for any δp. Hence, the solution to the minimization problem (2) is a solution to (1).

In fact, there exists another minimization problem which gives rise to the Poisson equation. In order to present it, we have first to rewrite the Poisson equation in a *mixed* form. The mixed form is obtained by introducing the flux function $\mathbf{v} = -\nabla p$ and rewriting (1) as

$$\mathbf{v} = -\nabla p, \tag{4a}$$

$$\nabla \cdot \mathbf{v} = h. \tag{4b}$$

The equations (4) are directly related to the minimization problem

$$\min \frac{1}{2} \int_\Omega |\mathbf{v}|^2 \, dx \tag{5a}$$

subject to the constraint

$$\nabla \cdot \mathbf{v} = h. \tag{5b}$$

The integral in (5a) corresponds to an energy and the minimization problem (5) can be interpreted as minimizing the energy given a *conservation* constraint (5b) (when

Date: November 1, 2015.

$h = 0$, the relation $\nabla \cdot \mathbf{v} = 0$ corresponds to the conservation of the quantity which is transported by the flux $\mathbf{v}$). In order to see the relation between the problem (4) and (5), we will use Lagrange multipliers, which is the standard approach to deal with optimization problem with constraint. We recall some basic facts on Lagrange multipliers in finite dimension. We consider the minimization problem

$$\min f(x) \tag{6a}$$

subject to the constraint

$$g(x) = 0. \tag{6b}$$

Here, $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ with typically $m < n$. A necessary condition for x to be a solution to (6) is obtained by introducing the Lagrange multiplier $\lambda \in \mathbb{R}^m$ and the augmented Lagrangian function

$$L(x, \lambda) = f(x) - \lambda^T g(x). \tag{7}$$

Then, if x is a solution to (6), there exists $\lambda \in \mathbb{R}^m$ such that (x, λ) is a critical point of L, that is,

$$\frac{\partial L}{\partial x}(x, \lambda) = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda}(x, \lambda) = 0. \tag{8}$$

The conditions (8) yields

$$\nabla f(x) = Dg(x)^T \lambda \quad \text{and} \quad g(x) = 0. \tag{9}$$

We use now the same approach for the infinite dimensional minimization problem (5). We introduce the Lagrange multiplier $p(x)$, which is now a function, and form the augmented Lagrangian functional

$$L(\mathbf{v}, p) = \frac{1}{2} \int_\Omega |\mathbf{v}|^2 \, dx - \int_\Omega p(x)(\nabla \cdot \mathbf{v} - h(x)) \, dx \tag{10}$$

We compute the variation of L with respect to a perturbation $\delta\mathbf{v}$. The boundary condition is $(\mathbf{v} + \varepsilon\delta\mathbf{v}) \cdot \mathbf{n} = 0$ for all ε, which implies $\delta\mathbf{v} \cdot \mathbf{n} = 0$. We have

$$\begin{aligned} L(\mathbf{v} + \varepsilon\delta\mathbf{v}, p) &= \frac{1}{2} \int_\Omega |\mathbf{v} + \varepsilon\delta\mathbf{v}|^2 \, dx - \int_\Omega p, \nabla \cdot (\mathbf{v} + \varepsilon\delta\mathbf{v} - h) \, dx \\ &= L(\mathbf{v}, p) + \varepsilon \int_\Omega \mathbf{v} \cdot \delta\mathbf{v} \, dx - \varepsilon \int_\Omega p \, \nabla \cdot \delta\mathbf{v} \, dx + o(\varepsilon) \\ &= L(\mathbf{v}, p) + \varepsilon \int_\Omega (\mathbf{v} + \nabla p) \cdot \delta\mathbf{v} \, dx + o(\varepsilon), \end{aligned}$$

after integration by part. A critical point $(\mathbf{v}, p)$ must therefore satisfy $\int_\Omega (\mathbf{v} + \nabla p) \cdot \delta\mathbf{v} \, dx = 0$ for all perturbation function $\delta\mathbf{v}$ and therefore, we recover $\mathbf{v} = -\nabla p$, that is, (4a). In the literature ([2]), the minimization problems (5) and (2) are called primal and dual problems, respectively.

2. Weak form and discretization

We consider the mixed formulation (4) and introduce the weak formulation. We have that $(\mathbf{v}, p)$ solves (4) if

$$\int_\Omega \mathbf{v} \cdot \mathbf{u} \, dx + \int_\Omega \nabla p \cdot \mathbf{u} \, dx = 0, \qquad \text{for all } \mathbf{u}, \tag{11a}$$

$$\int_\Omega (\nabla \cdot \mathbf{v}) q \, dx = \int_\Omega (\nabla \cdot h) q \, dx, \quad \text{for all } q. \tag{11b}$$

We can introduce the bilinear forms

$$b(\mathbf{v}, \mathbf{u}) = \int_\Omega \mathbf{v} \cdot \mathbf{u} \, dx \quad \text{and} \quad c(\mathbf{u}, p) = \int_\Omega \nabla p \cdot \mathbf{u} \, dx \tag{12}$$

and the equations (11) can be rewritten as

$$b(\mathbf{v}, \mathbf{u}) + c(\mathbf{u}, p) = 0, \qquad \text{for all } \mathbf{u}, \tag{13a}$$

$$c(\mathbf{v}, q) = e(q) \quad \text{for all } q. \tag{13b}$$

for a linear form e which is defined from the source function h. In order to simplify the presentation, I have been careful not to introduce any functional space. Of course, they are primordial in any analysis of the equations. In a finite element method, we use consistent discretization of the functions entering (13) and the discrete equations takes the form

$$\begin{bmatrix} B & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ E \end{bmatrix}$$

where the matrix B and C are the discrete representations of the bilinear forms b and c (we still denote by $\mathbf{u}$ and p the finite dimensional unknown).

3. Mimetic finite difference

Let us consider a partition of Ω. We denote by $\mathcal{T}$ the set of cells and $\mathcal{F}$ the set of faces which delimit the cells. We can rewrite the minimization problem (5) as finding the functions $\mathbf{v}_K$ in each cell K which minimize

$$\min \sum_{K\in\mathcal{T}} \frac{1}{2} \int_K |\mathbf{v}_K(x)|^2 \, dx \tag{14a}$$

subject to the constraints

$$\nabla \cdot \mathbf{v}_K(x) = h_K(x), \tag{14b}$$

for $x \in K$ and all $K \in \mathcal{T}$ and

$$\mathbf{v}_K(x) \cdot \mathbf{n}_{K,f} + \mathbf{v}_{K'}(x) \cdot \mathbf{n}_{K',f} = 0, \tag{14c}$$

for $x \in f$ and all $f \in \mathcal{F}_{\text{int}}$ with $f = K \cap K'$. Here, $\mathcal{F}_{\text{int}}$ denote the set of internal faces. For a given cell K, we denote by $\mathcal{F}_K$ the set of faces that belong to the cell K. Equation (14c) expresses the continuity of the flux on the internal faces, the normal $\mathbf{n}_{K,f}$ is the outer normal of f with respect to the cell K, so that $\mathbf{n}_{K,f} = -\mathbf{n}_{K',f}$. Comparing (14) to (5), we observe that the minimization problem on one side has been relaxed as we have now a set of functions as variables ($\mathbf{v}_K$) instead of a single function $\mathbf{v}$ (more degrees of freedom). On the other side, we have the extra constraint given by (14c). After introducing the Lagrange multiplier function $\pi(x)$ which takes value on the faces, the augmented Lagrangian for the problem (14) can be written as

$$L(\mathbf{v}, p, \pi) = \sum_{K\in\mathcal{T}} \left(\frac{1}{2} \int_K |\mathbf{v}_K(x)|^2 \, dx - \int_K p_K(x)(\nabla \cdot \mathbf{v}_K - h_K) \, dx \right) + \sum_{f\in\mathcal{F}_{\text{int}}} \int_f \pi_f(x)(\mathbf{v}_K(x) \cdot \mathbf{n}_{K,f} + \mathbf{v}_{K'}(x) \cdot \mathbf{n}_{K',f}) \, dx \tag{15}$$

Let us consider the variation of L with respect to v_K. It yields

$$\int_K \mathbf{v}_K(x) \cdot \mathbf{u}_K \, dx - \int_K p_K \nabla \cdot \mathbf{u}_K \, dx + \sum_{f\in\mathcal{F}_K^1} \int_f \pi_f(x) \mathbf{u}_K(x) \cdot \mathbf{n}_{K,f} \, dx = 0, \tag{16}$$

for any function $\mathbf{u}_K$, where $\mathcal{F}_K^1 = \mathcal{F}_K \cap \mathcal{F}_{\text{int}}$. After integration by part, (16) yields

$$\int_K (\mathbf{v}_K(x) + \nabla p_K \cdot) \mathbf{u}_K \, dx + \sum_{f\in\mathcal{F}_K^1} \int_f (\pi_f(x) - p_K(x)) \mathbf{u}_K(x) \cdot \mathbf{n}_{K,f} \, dx = 0. \tag{17}$$

By taking $\mathbf{u}_K$ with compact support in K, we get that

$$\mathbf{v}_K = -\nabla p_K, \tag{18}$$

as the last integral in (16) cancels. We plug (18) into (17) and, as $\mathbf{u}_K$ can be chose arbitrarily, we get

$$\sum_{f\in\mathcal{F}^1_K} \int_f (\pi_f(x) - p_K(x))\mathbf{u}_K(x)\cdot\mathbf{n}_{K,f}\,dx = 0. \tag{19}$$

Therefore, the Lagrange multiplier functions π_f in fact correspond to the restriction of $p_K(x)$ on the faces.

In a mimetic finite difference approach, we discretize these equations in the following way. The discrete unknown variables are chosen to be the integrated flux value $v_{K,f}$ on each side of each face,

$$v_{K,f} \approx \int_f \mathbf{v}_K(x)\cdot\mathbf{n}_{K,f}\,dx$$

The mimetic method is a finite volume method so that the divergence operator is approximated by the sum of all the fluxes on the faces of a given cell,

$$\mathtt{div}_K(v_K) = \sum_{f\in\mathcal{F}^1_K} v_{K,f}. \tag{20}$$

It remains to approximate the energy functional on each cell K. Given our degrees of freedom (face fluxes), it is naturally done by introducing a symmetric positive matrix B^K such that

$$\int_K \mathbf{v}_K(x)\cdot\mathbf{u}_K(x)\,dx \approx \sum_{f\in\mathcal{F}^1_K} v_{K,f}B^K_{f,f'}u_{K,f'}. \tag{21}$$

The method will be called *mimetic* if

- B^K is invertible (stability condition),
- B^K is exact for constant fluxes (consistency condition).

This later statement means that, when one of the flux is constant, say $\mathbf{v}_K$, the approximation (21) becomes exact, that is, we have

$$\int_K \mathbf{v}_K\cdot\mathbf{u}_K(x)\,dx = \sum_{f\in\mathcal{F}^1_K} v_{K,f}B^K_{f,f'}u_{K,f'}. \tag{22}$$

for any constant vector $\mathbf{v}_K$ and for any function $\mathbf{u}_K(x)$ and, in (22),

$$v_{K,f'} = \int_f \mathbf{v}_K\cdot\mathbf{n}_{K,f}\,dx = |f|\;\mathbf{v}_K\cdot\mathbf{n}_{K,f} \quad\text{ and }\quad u_{K,f'} = \int_f \mathbf{u}_K(x)\cdot\mathbf{n}_{K,f}\,dx,$$

where $|f|$ denotes the area of the face f. Then, it can be proven that any mimetic method lead to a convergent scheme, see [3]. Given a face f, the neighboring cells of the f are ordered and denoted by $K_{1,f}$ and $K_{2,f}$. We introduce the operator $R_{1,f}$ and $R_{2,f}$ which, when applied to v, return the values of $v_{K_1,f}$ and $v_{K_2,f}$. Finally, the discrete version of the minimization problem can be rewritten as

$$\min \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{f,f'\in\mathcal{F}^1_K} v_{K,f}B^K_{f,f'}v_{K,f'} \tag{23a}$$

subject to the constraint

$$\mathtt{div}_K(v) = h \tag{23b}$$

for all $K \in \mathcal{T}$ and

$$R_{1,f}(v) + R_{2,f}(v) = 0 \tag{23c}$$

for all $f \in \mathcal{F}$. The augmented Lagrangian for (23) is given by

$$L(v, p, \pi) = \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{f, f' \in \mathcal{F}_K^1} v_{K,f} B_{f,f'}^K v_{K,f'} - \sum_{K \in \mathcal{T}} p_K(\mathtt{div}_K(v) - h_K) + \sum_{f \in \mathcal{F}} \pi_f(R_{1,f}(v) + R_{2,f}(v)). \tag{24}$$

The discrete equations are then obtained by computing the critical points of L. We end up with a system of the form

$$\begin{bmatrix} B & C & D \\ C^T & 0 & 0 \\ D^T & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ \pi \end{bmatrix} = \begin{bmatrix} 0 \\ h \\ 0 \end{bmatrix}. \tag{25}$$

Note that the sparsity and the symmetry of the matrix in (25) follows directly from the fact that the variables p and π are Lagrangian multipliers. Moreover, since B^K are positive symmetric definite, we have that B inherits the same properties.

4. The Dirichlet boundary condition

Until now, we have used Neumann boundary condition. Let us consider the Dirichlet condition

$$p(x) = p_{\text{dir}}(x) \text{ for } x \in \Gamma_{\text{dir}} \tag{26}$$

where $\Gamma_{\text{dir}} \subset \partial\Omega$, for a given function p_{dir}. On $\Gamma_{\text{neum}} = \partial\Omega \setminus \Gamma_{\text{dir}}$, we still impose Neumann boundary conditions, $\mathbf{v} \cdot \mathbf{n} = 0$. To introduce this case using a variational approach it is convenient to look at p as a pressure. Darcy's law gives us that $\mathbf{v}$ is proportional to the gradient of pressure and, for simplicity, we set $\mathbf{v} = \nabla p$. The condition $\nabla \cdot \mathbf{v} = 0$ will then be equivalent to the conservation of mass in the absence of source. The energy $\frac{1}{2} \int_\Omega |v|^2 \, dx$ corresponds to the dissipative energy. A pressure given at the boundary allows for a flux in and out of the domain. This pressure force exerts some work which must be included in the energy balance of the system. The power exerted is

$$\int_{\Gamma_{\text{dir}}} p_{\text{dir}}(x) \mathbf{v}(x).\mathbf{n} \, dx \tag{27}$$

The governing equations are thus obtained deriving the necessary condition for obtaining the solution to the minimization problem

$$\min \left(\frac{1}{2} \int_\Omega |\mathbf{v}|^2 \, dx - \int_{\Gamma_{\text{dir}}} p_{\text{dir}}(x) \mathbf{v}(x) \cdot \mathbf{n} \, dx \right) \tag{28a}$$

subject to the constraint

$$\nabla \cdot \mathbf{v} = h. \tag{28b}$$

The augmented Lagrangian functional for (28) is

$$L(\mathbf{v}, p) = \frac{1}{2} \int_\Omega |\mathbf{v}|^2 \, dx - \int_{\Gamma_{\text{dir}}} p_{\text{dir}}(x) \mathbf{v}(x) \cdot \mathbf{n} \, dx - \int_\Omega p(x) \nabla \cdot (\mathbf{v}(x)) \, dx$$

We consider the variation with respect to $\mathbf{v}$ given by $\mathbf{v} + \varepsilon \delta \mathbf{v}$. From the boundary condition, we get that $\mathbf{v} \cdot \mathbf{n} = 0$ for all $x \in \Gamma_{\text{neum}}$, which implies, as we have seen before, implies a constraint on the perturbation functions, namely $\delta \mathbf{v} \cdot \mathbf{n} = 0$ on

Γ_{neum}. However, we can relax this constraint by adding yet another Lagrangian multiplier, which we denote p_{neum}, and consider the augmented Lagrangian

$$L(\mathbf{v},p) = \frac{1}{2}\int_\Omega |\mathbf{v}|^2 \, dx - \int_{\Gamma_{\mathrm{dir}}} p_{\mathrm{dir}}(x)\mathbf{v}(x)\cdot\mathbf{n}\,dx - \int_{\Gamma_{\mathrm{neum}}} p_{\mathrm{neum}}(x)\mathbf{v}(x)\cdot\mathbf{n}\,dx - \int_\Omega p(x)\nabla\cdot(\mathbf{v}(x))\,dx \tag{29}$$

The variation with respect to $\mathbf{v}$ gives us

$$\begin{aligned} L(\mathbf{v}+\varepsilon\delta\mathbf{v},p) &= L(\mathbf{v},p) + \varepsilon\Big(\int_\Omega \mathbf{v}\cdot\delta\mathbf{v}\,dx - \int_{\Gamma_{\mathrm{dir}}} p_{\mathrm{dir}}(x)\delta\mathbf{v}(x)\cdot\mathbf{n}\,dx \\ &\quad - \int_{\Gamma_{\mathrm{neum}}} p_{\mathrm{neum}}(x)\delta\mathbf{v}(x)\cdot\mathbf{n}\,dx - \int_\Omega p(x)\nabla\cdot(\delta\mathbf{v}(x))\,dx\Big) + o(\varepsilon) \\ &= L(\mathbf{v},p) + \varepsilon\Big(\int_\Omega (\mathbf{v}+\nabla p)\cdot\delta\mathbf{v}\,dx + \int_{\Gamma_{\mathrm{dir}}} (p(x)-p_{\mathrm{dir}}(x))\delta\mathbf{v}\cdot\mathbf{n}\,dx \\ &\quad + \int_{\Gamma_{\mathrm{neum}}} (p(x)-p_{\mathrm{neum}}(x))\delta\mathbf{v}\cdot\mathbf{n}\,dx\Big) + o(\varepsilon), \end{aligned}$$

after integration by part. Hence, we must have

$$\int_\Omega (\mathbf{v}+\nabla p)\cdot\delta\mathbf{v}\,dx + \int_{\Gamma_{\mathrm{dir}}} (p-p_{\mathrm{dir}})\delta\mathbf{v}\cdot\mathbf{n}\,dx + \int_{\Gamma_{\mathrm{neum}}} (p-p_{\mathrm{neum}})\delta\mathbf{v}\cdot\mathbf{n}\,dx = 0 \tag{30}$$

for all $\delta\mathbf{v}$. First, we choose $\delta\mathbf{v}$ with compact support and we get $\mathbf{v} = -\nabla p$ and the first integral in (30) vanishes. Then, we can choose $\delta\mathbf{v}\cdot\mathbf{n}$ arbitrarily and get that $p = p_{\mathrm{dir}}$ on Γ_{dir} and $p = p_{\mathrm{neum}}$ on Γ_{neum}. Finally, we obtain the equations

$$\nabla\cdot\mathbf{v} = h \quad\text{and}\quad \mathbf{v} = -\nabla p \quad \text{in } \Omega, \tag{31}$$

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\setminus\Gamma_{\mathrm{dir}}, \tag{32}$$

$$p = p_{\mathrm{dir}} \quad \text{on } \Gamma_{\mathrm{dir}} \tag{33}$$

and $p_{\mathrm{neum}} = p$ on Γ_{neum} but this last identity is only interesting when we will look at the discretization. For the discrete equation the augmented Lagrangian is given by

$$\begin{aligned} L(v,p,\pi) &= \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{f,f'\in\mathcal{F}_K^2} v_{K,f}B^K_{f,f'}v_{K,f'} - \sum_{K\in\mathcal{T}} p_K(\mathtt{div}_K(v) - h_K) \\ &\quad - \sum_{\substack{f\in\mathcal{F}_{\mathrm{dir}}\\(K=\mathcal{T}_f)}} p_{\mathrm{dir},f}v_{K,f} - \sum_{\substack{f\in\mathcal{F}_{\mathrm{neum}}\\(K=\mathcal{T}_f)}} p_{\mathrm{neum},f}v_{K,f} + \sum_{f\in\mathcal{F}_{\mathrm{int}}} \pi_f(R_{1,f}(v) + R_{2,f}(v)). \end{aligned} \tag{34}$$

Here, $\mathcal{T}_f$ denotes the neighboring cells of the face f. In the third and fourth sum above, since f corresponds to a external face, the set $\mathcal{T}_f$ reduces to a single element. Note that in (34), p_{dir} is a known function while p_{neum} is an unknown variables, which will be equal to the pressure on the faces where the Neumann condition holds.

5. Schur-Complement

We reduce the system using Schur-complements. We decouple p and π from v using the following manipulation,

$$\begin{bmatrix} B^{-1} & 0 & 0 \\ -C^TB^{-1} & \mathrm{Id} & 0 \\ -D^TB^{-1} & 0 & \mathrm{Id} \end{bmatrix}\begin{bmatrix} B & C & D \\ C^T & 0 & 0 \\ D^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Id} & B^{-1}C & B^{-1}D \\ 0 & -C^TB^{-1}C & -C^TB^{-1}D \\ 0 & -D^TB^{-1}C & -D^TB^{-1}D \end{bmatrix}$$

Let us introduce Q, H, F, E defined as follows

$$Q = \begin{bmatrix} H & F \\ F^T & E \end{bmatrix} = \begin{bmatrix} C^T B^{-1} C & C^T B^{-1} D \\ D^T B^{-1} C & D^T B^{-1} D \end{bmatrix}.$$

Note that Q, H and E are by definition symmetric because B is symmetric. We can rewrite Q as

$$Q = \begin{bmatrix} C^T \\ D^T \end{bmatrix} B^{-1} \begin{bmatrix} C & D \end{bmatrix},$$

which shows that the matrix Q is positive. We decouple π from p by the following manipulation,

$$\begin{bmatrix} \mathrm{Id} & 0 \\ -F^T H^{-1} & \mathrm{Id} \end{bmatrix} Q = \begin{bmatrix} \mathrm{Id} & 0 \\ -F^T H^{-1} & \mathrm{Id} \end{bmatrix} \begin{bmatrix} H & F \\ F^T & E \end{bmatrix} = \begin{bmatrix} H & F \\ 0 & S \end{bmatrix},$$

where

$$S = E - F^T H^{-1} F.$$

The matrix S is by definition symmetric. We have

$$\begin{bmatrix} \mathrm{Id} & 0 \\ -F^T H^{-1} & \mathrm{Id} \end{bmatrix} Q \begin{bmatrix} \mathrm{Id} & -H^{-1} F \\ 0 & \mathrm{Id} \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & S \end{bmatrix}.$$

Hence the positivity of Q implies the positivity of S. The initial system (25) can thus be solved by first computing π given as the solution to

$$S\pi = F^T H^{-1} h.$$

Then p is computed by solving

$$Hp + F\pi = -h$$

and, finally, v is computed by solving

$$Bv + Cp + D\pi = 0.$$

6. Summary

In the Matlab code based on MRST which is given on the website of the course, the matrix B, C and D which appear in (25) are assembled. Therefore, the only thing you have to understand in order to use the code is the structure of the data, that is, the significance of the variable v, p and π. To do so, get acquainted with the grid structure, by typing the command `help grid_structure` and looking at [4, section 3.4]. Moreover, once the solution (v, p, π) of (25) is computed, we have the following correspondences with respect to the solution $p(x)$ of the initial problem (1),

$$v_{K,f} \approx \int_f \nabla p(x) \cdot \mathbf{n}_{K,f}\, dx, \qquad p_K \approx \frac{1}{|K|} \int_K p(x)\, dx, \qquad \pi_f \approx \frac{1}{|f|} \int_f p(x)\, dx.$$

References

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