TMA4195 - MATHEMATICAL MODELING (FALL 2015). PROJECT DESCRIPTION

BØLGEN

1. Introduction

Rocks detaching from mountain's steep faces along the fjords and sliding into the water can create devastating tsunami waves. Such events have taken place in Norway at Loen (in 1905 and 1936) and Tafjord (in 1934), see [1, 2, 3]. An unstable slope has been identified at Åkneset and it is now the subject of extensive monitoring, see [4]. The slide volume is estimated to 54 million m^3, with its lowest and highest points located respectively at 150 m and 900 m above the sea level, see Figure 1 and [5] for a detailed description of this site.

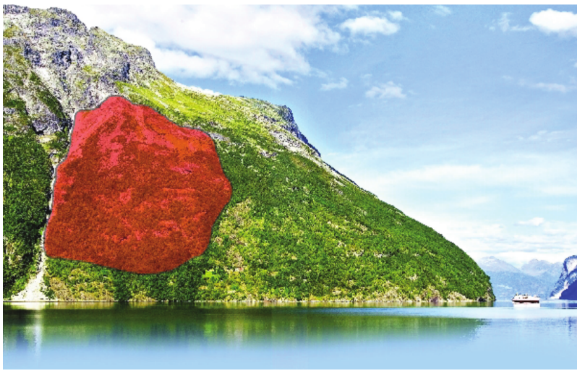

Figure 1. Plot of the unstable slope at Åkneset, from [6]

The goal of this project is to set up simple models for tsunami waves generated by rock slides. We will try to address the three major phases in the development of a tsunami:

(1) The creation of the tsunami wave, when the kinetic energy of a falling rock is transferred to the water (section 4).
(2) The propagation of the wave in the parts of the fjord where the bottom remains more or less at the same elevation (sections 2 and 3).
(3) The run-up of the wave when it reaches the end of the fjord and the height of the wave starts increasing at the same time that its speed decreases (section 5).

In a context of a tsunami event close to populated area, two characteristics of the tsunami will directly determine the actions that have to be taken:

(1) How long does it take for the wave to reach the populated areas (In the film Bølgen [7], it takes 10 minutes ...)?

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(2) How high will the wave be when it reaches the shore?

For this two-weeks project, you are invited to answer to these questions. The modeling of tsunami waves represents a whole field of active research and you are not expected to come with state-of-the-art solutions. You are expected to attack it at your level, using all the tools you have been equipped with, as *mathematical modelers*, and which roughly cover the following three categories

(1) Modeling tools in order to describe the problem, derive the modeling equations and finally simplify them by only retaining the processes that matter for a given application (questions of type *mod*).
(2) Analytical tools in order to solve simple equations and, in that way, gain more insight into the governing equations (questions of type *ana*).
(3) Numerical tools in order to set up and implement numerical methods which will enable you to treat more realistic cases (questions of type *num*).

2. The model equations

We have a fluid with a density $\rho(t, \mathbf{x})$ and a velocity field $\mathbf{v}(t, \mathbf{x})$. The conservation of mass equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.1}$$

Question (mod): *Recall how Equation* (2.1) *is obtained.*

For a point with mass m, Newton second law of motion states that the momentum is preserved in the absence of external forces and, if there are external forces, the rate of change in momentum is equal to the sum of the external forces, that is,

$$\frac{d}{dt}(mv) = \sum_i F_i. \tag{2.2}$$

Given a fluid volume $\Omega(t)$ which contains the same water particles, the exterior forces are due to the pressure forces $p(t, \mathbf{x})$ exerted by the particle outside $\Omega(t)$ and gravitation. In this case, Newton second law of motion can be rewritten as

$$\frac{d}{dt}\int_{\Omega(t)} \rho \mathbf{v}\, dV = -\int_{\partial\Omega(t)} p\mathbf{n}\, dS + \int_{\Omega(t)} \rho \mathbf{g}\, dV \tag{2.3}$$

Using Reynolds transport theorem and (2.1) we can obtain from (2.3) the equation of conservation of momentum, written in convective form,

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{g}. \tag{2.4}$$

Question (mod): *Derive* (2.4) *directly, without using Reynolds transport theorem and* (2.1)*, by dividing with* $\int_{\Omega(t)} \rho\, dV$ *and letting the volume* $\Omega(t)$ *tend to zero.*

The geometry of the problem is depicted in Figure 2. The water surface at time t is given by $z = \eta(t, x, y)$ and we assume that $z = 0$ is the water surface at rest so that $h(x, y)$ is defined as the depth. The governing equations are thus the conservation equations for mass and momentum in the evolving domain defined by

$$\Omega(t) = \left\{(x, y, z) \in \mathbb{R}^3 \mid -h(x, y) < z < \eta(t, x, y)\right\} \tag{2.5}$$

Question (mod): *What are the boundary conditions?*

To simplify the equations, we assume that the flow is irrotational, that is,

$$\nabla \times \mathbf{v} = 0. \tag{2.6}$$

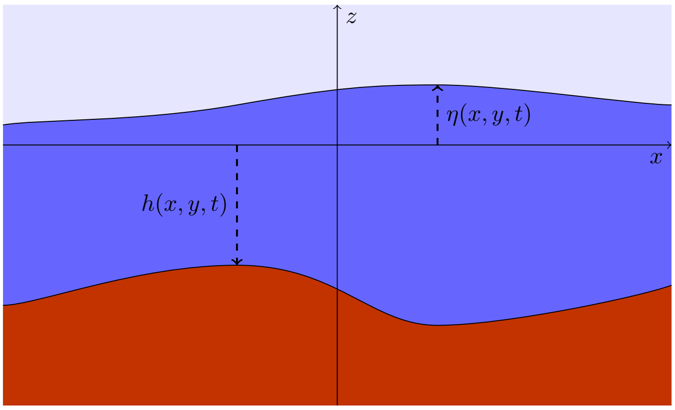

FIGURE 2. Plot of a water segment

In this case, we have that $\mathbf{v}$ can be derived from a potential, that is, there exists a scalar function ϕ such that

$$\mathbf{v} = \nabla\phi. \tag{2.7}$$

Question (mod): *Explain why* (2.6) *implies* (2.7).

We assume that water is incompressible. Then, we obtain the following equations for ϕ,

$$\Delta\phi = 0, \tag{2.8a}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p - p_{\text{atm}}}{\rho} + gz = 0. \tag{2.8b}$$

Taking into account the boundary conditions, equation (2.8b) yields

$$\nabla\phi \cdot \mathbf{n} = 0 \text{ for } z + h(x, y) = 0, \tag{2.9a}$$

$$0 = \eta_t + \nabla\phi \cdot [\eta_x, \eta_y, -1] \quad \text{ and} \tag{2.9b}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + g\eta = 0 \tag{2.9c}$$

for $z = \eta(x, y)$.

Question (mod): *Explain how the equations* (2.8) *and* (2.9) *are obtained.*

We note that (2.8b) gives us an explicit expression for the pressure as a function of the other unknown ϕ and η. Moreover, the equations (2.8a) and (2.9) provide us with governing equations for ϕ and η only, which are independent of pressure. Hence, to solve the full system, we can first solve (2.8a) and (2.9) to obtain ϕ and η and then recover the pressure by using (2.8b). Let us assume that we have implemented numerically a Laplace solver which can handle generically domain of the form given by (2.5) for any function η (for a fixed time t).

Question (num): *Describe how you could use such Laplace solver to solve the governing equations* (2.8a) *and* (2.9).

Question (num): *Implement the method you have just described and present the results of some simulations. You can either implement the Laplace solver from scratch or build up on the code using MRST ([8]), which is available on the course website.*

3. Reduction of a linear model

Question (mod): *Undimensionalize the equations* (2.8) *and* (2.9).

Hint : Can you relate the expression you obtained to the following one

$$\mu(\phi_{xx} + \phi_{yy}) + \phi_{zz} = 0 \tag{3.1a}$$

with boundary conditions

$$\mu(\phi_x h_x + \phi_y h_y) + \phi_z = 0 \tag{3.1b}$$

for $z + h = 0$,

$$\mu\eta_t + \varepsilon\mu(\phi_x h_x + \phi_y h_y) = \phi_z \quad \text{and} \tag{3.1c}$$

$$\phi_t + \frac{1}{2}\varepsilon(\phi_x^2 + \phi_y^2) + \frac{1}{2}\frac{\varepsilon}{\mu}\phi_z^2 + \eta = 0 \tag{3.1d}$$

for $z = \varepsilon\eta(x, y)$ □

Now η that the equations have been undimensionalized, we consider the two-dimensional case by assuming that the solution is invariant in the y-direction. When we neglect the non-linear terms, the equations reduced to

$$\mu\phi_{xx} + \phi_{zz} = 0 \tag{3.2a}$$

for $x \in \mathbb{R}$ and $z \in [0, 1]$, with the boundary conditions

$$\phi_z = 0 \tag{3.2b}$$

for $z + 1 = 0$ and

$$\mu\phi_{tt} + \phi_z = 0 \tag{3.2c}$$

for $z = 0$

Question (mod): *Explain the derivation of* (3.2). *What does μ represent? What happened to η?*

Question (ana): *The equations* (3.2) *are linear. Use the superposition principle to compute the solutions to* (3.2). *You may first assume that $x \in [0, 1]$. Try to obtain a dispersion relation.*

4. Generation of the wave

In the previous sections, the initial conditions were not mentioned. To determine them, let us consider an approach based on energy preservation. When falling into the water, energy is transferred from the rock into water, in form of heat and kinetic energy.

Question (mod): *What is the total energy of a rock of mass m, velocity m at height z?*

When the friction forces balance the gravitation force, a body in free fall reaches a constant velocity which we denote v_∞.

Question (mod): *List the parameters from which v_∞ depends. Use dimensional analysis to get a simple expression relating v_∞ with those parameters.*

Question (mod): *Try to estimate the kinetic energy that can be transferred to water. How can use this value to set up an initial condition for the model equations* (2.1) *and* (2.4).

Question (ana, num): *Apply the initial condition derived in the previous questions to the numerical solver and the analytical solutions derived in the previous sections.*

5. Inundation

The shallow water equations, which are given by

$$\eta_t + (u(\varepsilon\eta + h))_x = 0, \tag{5.1a}$$

$$u_t + \left(\frac{1}{2}\varepsilon u^2 + \eta\right)_x = 0, \tag{5.1b}$$

are a good approximation when the water depth is much smaller than the typical wavelength.

Question (mod): *Using an asymptotic expansion with respect to the ratio depth over typical wavelength, derive the shallow-water equations from* (2.8) *and* (2.9). *What does* ε *represent?*

Hint : Show that the adimensional variable μ in (3.1) corresponds to the ratio typical depth over typical wave length. Then, insert expansions of the form

$$\phi(x,z) = \phi_0(x,z) + \mu\phi_1(x,z) + o(\mu),$$
$$\eta(x,z) = \eta_0(x,z) + \mu\eta_1(x,z) + o(\mu),$$

in (3.1). From the expressions you obtain, equal the terms of same orders in μ. Then, you obtain a set of equations which should hopefully leads you to (5.1). Do not forget to justify the approach. □

Let us first consider a flat bottom, that is $h_x = 0$. Then, the system can be rewritten as a system conservation laws

$$u_t + \left(\frac{u^2}{2} + \eta\right)_x = 0, \tag{5.2a}$$

$$\eta_t + (u\eta)_x = 0. \tag{5.2b}$$

In (5.2), in order to simplify the expressions, we set $\varepsilon = 1$ and let η be the total height, that is, $\varepsilon\eta + h$ in the notations of Figure 2. We want to see if there exist non-trivial functions $V(u,\eta)$ and $W(u,\eta)$ of u and η such that the system (5.2) decouples, that is, (5.2) becomes equivalent to

$$V_t + \lambda_1 V_x = 0, \tag{5.3a}$$

$$W_t + \lambda_2 W_x = 0, \tag{5.3b}$$

where λ_1 and λ_2 are also function of u and η.

Question (ana): *From* (5.3)*, figure out necessary conditions that the functions* V*,* W*,* λ_1 *and* λ_2 *must satisfy. Compute them.*

Hint : Check your result: Once you have obtained V, compute V_t and V_x by using the chain rule and then use the equations (5.2) to check that (5.3a) is satisfied. Can you check that, for example, that $V = u + 2\sqrt{\eta}$ and $W = u - 2\sqrt{\eta}$ are solutions to (5.3) for some λ_1 and λ_2 ? □

The functions V and W are called Riemann invariants.

Question (num): *Use* (5.3) *to set up a simple solver based on characteristics curves.*

Let us now consider the case of a bottom with constant slope, that is, $h_x = a$ for $a < 0$. For the same Riemann invariants V, W and functions λ_1 and λ_2 as in (5.3), we get that the shallow water equation is now equivalent to

$$V_t + \lambda_1 V_x = a, \tag{5.4a}$$

$$W_t + \lambda_2 W_x = a, \tag{5.4b}$$

Question (ana): *Derive the equations* (5.4).

We observe that if, initially $W_x(0, x) = 0$ for all x, then $W_x(t, x) = 0$ for all t, x.

Question (ana): *Prove this observation and use it to reduce the system* (5.4) *to a scalar equation. Solve this first order scalar partial differential equation using the method of characteristics. Use initial data that will show the build-up of the wave, that is, when the wave slows down and increases height as it reaches the shore.*

When $h_x \neq 0$, the system of equations (5.1) is called a *balanced law*, see [9]. When applied to a balance law, a numerical methods is said to be *well-balanced* if it computes the equilibrium state exactly. Numerical schemes that are not well-balanced usually suffer from instability. In [10], a state-of-the-art method to solve the shallow water wave equation is described.

Question (num): *Implement the numerical method described [10] and run simulations that illustrates the build-up of the wave.*

References

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