TMA4195 Matematisk modellering

Solution of First Order Quasi-linear Partial Differential Equations - A First Aid Course

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This note gives a short introduction to solution of first order partial differential equations (PDEs) that occur in connection with models based on conservation principles. It aims at students with only calculus background.

1 What Does the Title Mean?

The title contains several words which may be unknown:

- *First Order* = only first order derivatives occur in the equation
- *Quasi-linear* = the equation is linear in the first order derivatives
- *Partial* = there is more than one independent variable

2 Equations and Solutions

The theory below is illustrated for one variable z dependent on two independent variables x and y. Equations with more independent variables are solved in a similar way.

PDEs are divided into several classes, and for an equation to belong to the class in the title, it is necessary that it can be put into what we call the *normal form*. This means that the equation can be written

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}-R(x,y,z)=0. \quad (1)$$

Here P, Q and R are functions only of x, y, and z, and do not contain any derivatives. Note that $\partial z/\partial x$ and $\partial z/\partial y$ only occur in the first power, but there is no such limitation for z in P, Q and R. The reason for the minus in front of the third term will be clear below.

A *solution* of Eqn. 1 is a function,

$$z=f(x,y), \quad (2)$$

which satisfies the equation:

$$P(x, y, f(x, y)) \frac{\partial f(x, y)}{\partial x} + Q(x, y, f(x, y)) \frac{\partial f(x, y)}{\partial y} - R(x, y, f(x, y)) = 0. \qquad (3)$$

If we consider a regular coordinate system, $(x, y, z) \in \mathbb{R}^3$, the function $z = f(x, y)$ will define a *surface* in $\mathbb{R}^3$. Typically, finding a solution to Eqn. 1 means to find a function $f(x, y)$ fulfilling some addition conditions, e.g. having given values on some curve in the xy-plane. It will soon become clear that solving a PDE is radically different from solving ordinary diff. equations, although ordinary equations sometimes come up during the solution process.

From Calculus we probably remember that the vector

$$\mathbf{n} = \left[\frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y}, -1\right] \qquad (4)$$

is a *normal vector* (perpendicular) to the surface $z = f(x, y)$ in the point (x, y, z) (Try to derive this yourself if you do not know it).

A *vector field* $\mathbf{V}(x, y, z)$ in $\mathbb{R}^3$ is defined in terms of three functions making up the three components of the vector, say

$$\mathbf{V}(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]. \qquad (5)$$

A vector field defines a set of *stream lines* in space. Curves in space may be parametrized by a variable s and written as

$$\mathbf{r}(s) = [x(s), y(s), z(s)], s \in \mathbb{R}. \qquad (6)$$

The stream lines for the vector field $\mathbf{V}$ satisfy the following system of differential equations

$$\frac{d\mathbf{r}}{ds} = \mathbf{V}(x, y, z), \qquad (7)$$

or, written out,

$$\begin{aligned} \frac{dx}{ds} &= P(x, y, z), \\ \frac{dy}{ds} &= Q(x, y, z), \\ \frac{dz}{ds} &= R(x, y, z). \end{aligned} \qquad (8)$$

In general, one can set $\mathbf{r}(s_0) = \mathbf{r}_0 = [x_0, y_0, z_0]$ and solve the system 8 in order to find the stream line through $\mathbf{r}_0$. In the PDE literature, you often find Eqn. 8 written as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \qquad (9)$$

This means *exactly the same* and is nothing but a short way of writing Eqn. 8.

We now make an interesting observation:

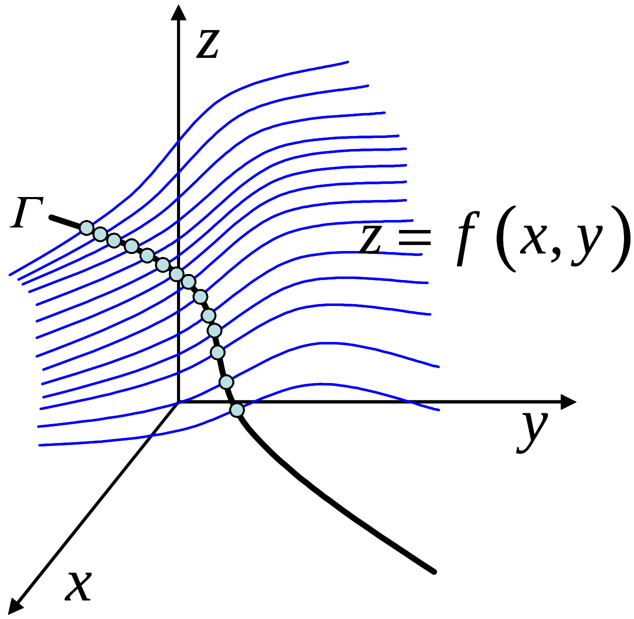

Figure 1: The characteristics through Γ define a solution to the Cauchy problem.

- *The normal vector to a solution of Eqn. 1 is perpendicular to the stream lines of the vector field* $\mathbf{V}$*, defined as in Eqn. 5 with* P*,* Q*, and* R *from Eqn. 1.*

This is quite obvious since

$$\mathbf{V} \cdot \mathbf{n} = [P, Q, R] \cdot \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right] = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} - R = 0! \tag{10}$$

In the PDE theory, the stream lines are called *characteristic curves*, or simply *characteristics*.

It is the important observation above that now makes it possible for us to solve the so-called *Cauchy problem*:

Given a curve Γ *in space. Find a function* $z = f(x, y)$ *that satisfies Eqn. 1 and is such that the curve* Γ *is contained in the surface defined by the solution.*

The Cauchy problem is the common name for such problems, resembling what we would call an initial value problem for ordinary diff. equations.

If we think in practical terms, is actually not so difficult to imagine how this could be carried out: For all points on Γ, we find the characteristic curves through the points. When we then move along Γ, the characteristics slice out a surface in space. By the way the surface is made, the normal vectors to the surface must be orthogonal to the characteristics. In other words, we have actually got the situation in Eqn. 10, and have found a solution to 1, as illustrated in Fig. 1.

When P, Q, and R are nice and reasonable functions, the solution of Eqn. 8 will be unique. This means that only one characteristic curve can pass through each point in space, or that two different characteristic curves in space can never collide. However, this does not prevent a solution, as the one on Fig. 1, from "folding over", meaning that one has two different z-values for each point (x, y). We shall see later that this complicates matters for real world problems where z has a physical meaning.

There are also two special situations where the characteristic method needs to be modified. The first is when Γ itself (or parts of it) is a characteristic curve. Where this is the case, a *unique* solution to the equation can not be obtained. The second possibility is when there exists functions $f(x, y)$ so that

$$\begin{aligned} P\left(x, y, f\left(x, y\right)\right) &= 0, \\ Q\left(x, y, f\left(x, y\right)\right) &= 0, \\ R\left(x, y, f\left(x, y\right)\right) &= 0. \end{aligned} \tag{11}$$

Such functions are called *singular solutions*, since they obviously satisfy Eqn. 1.

Even if this way of solving the equations may seem simple, it is quite implicit and not always so easy to carry out. Finally, even if both Γ and the characteristics are known, it may be difficult or even impossible to write the solution in the explicit form $z = f(x, y)$.

3 An Example

The following example is taken from Volume III of the classic calculus textbook "*Lærebok i matematisk analyse*" by R. Tambs Lyche, §282 (used at NTH for generations).

We are expected to find the solution of the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - 1 = 0 \tag{12}$$

passing through the space curve

$$\mathbf{r}\left(t\right) = -\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k},\ t \in \mathbb{R} \tag{13}$$

($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the respective axes).

The first step will be to determine the characteristics, that is, to solve

$$\begin{aligned} \frac{dx}{ds} &= P\left(x, y, z\right) = x, \\ \frac{dy}{ds} &= Q\left(x, y, z\right) = y, \\ \frac{dz}{ds} &= R\left(x, y, z\right) = 1. \end{aligned} \tag{14}$$

It is not difficult to solve these equations since they do not interfere with each other:

$$\begin{aligned} x\left(s\right) &= C_1 e^s, \\ y\left(s\right) &= C_2 e^s, \\ z\left(s\right) &= s + C_3. \end{aligned} \tag{15}$$

It remains to find expressions for the special characteristics that pass through the space curve, and thus eliminate the free constants C_1, C_2 and C_3 in Eqn. 15. There are several ways to proceed, and the following is somewhat simpler than the method used in the reference. Let us (without loss of generality) assume that the characteristics cross the curve for $s = 0$. We then obtain

$$\begin{aligned} x(0) &= C_1 = -1, \\ y(0) &= C_2 = 2t, \\ z(0) &= C_3 = t^2. \end{aligned}$$

The solution is thus the surface defined in *parametric* form as

$$\begin{aligned} x &= C_1 e^s = -e^s, \\ y &= C_2 e^s = 2te^s, \\ z &= s + C_3 = s + t^2, \end{aligned} \tag{16}$$

for the pair of parameters $(s, t) \in \mathbb{R}^2$.

In this special case it is actually also possible to eliminate s and t, and write z as a function of x and y. From the two first equations in Eqn. 16 we see that

$$s = \ln(-x), \tag{17}$$

$$t = -\frac{y}{2x}. \tag{18}$$

If this is inserted into the third equation we obtain

$$z = \ln(-x) + \left(\frac{y}{2x}\right)^2. \tag{19}$$

4 Recipe

1. Be sure that the PDE is written in the form

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} - R(x, y, z) = 0, \tag{20}$$

   with no derivatives in P, Q, and R. Do not forget the minus sign in front of R!

2. Determine the Γ-curve and put it in the parametric form

$$\mathbf{r}(s) = [x(s), y(s), z(s)], s \in \mathbb{R}.$$

   Often, it is possible to use x or y as the parameter, say, $x = x$, $y = g(x)$, $z = f(x)$.

3. Form the ODE system for the characteristic curves,

$$\begin{aligned} \frac{dx}{ds} &= P(x, y, z), \\ \frac{dy}{ds} &= Q(x, y, z), \\ \frac{dz}{ds} &= R(x, y, z), \end{aligned} \tag{21}$$

   and solve it by some standard method (In real life, this may have to be carried out numerically).

4. Determine the collection of characteristics passing through Γ by imposing appropriate initial conditions on the solution found in the previous point. Make sure this defines a surface, which, as in the example, may sometimes be reduced to $z = f(x, y)$.

GOOD LUCK!