Equilibrium points

1. **Equilibrium point:**
   A constant solution u_e of the problem (e.g. ODEs or PDEs)

2. **Stable** equilibrium point u_e:
   All solutions $u(t)$ starting near u_e, remain near u_e for all $t \geq 0$:
   $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $|u(0) - u_e| \leq \delta \Rightarrow |u(t) - u_e| < \varepsilon \quad \forall t > 0$

3. **Linear stability analysis** for u_e
   1. set solution $u = u_e + \tilde{u}$, $\quad |\tilde{u}| \ll 1$ small perturbation
   2. insert into equation, drop small(=non-linear) terms
      $\longrightarrow$ linear equation(s) for $\tilde{u}$ (= linearized equation(s))
   3. Check if all solutions of linearized equation(s) starting near 0
      (= all small perturbations) remain near.
      $\longrightarrow$ If yes (no): indicate that u_e is stable (unstable).

4. Over time all physical systems tend to be at their stable equilibrium solutions! (... always small disturbances ...)

Aggregation of Amoeba

Full discussion in Chapter 6 in *Logan: Applied Mathematics* (compendium)

Background:

Lack of food → amoeba produce attractant and aggregate.

Question:

Can onset of aggregation be caused by simple, uninteligent mechanism?

Model near onset of aggregation:

Physical quantities:

$a(x,t)$, $c(x,t)$ = amoeba, attractant densities; parameters

Conservation + attraction + diffusion + production:

$$(1) \qquad a_t = \frac{\partial}{\partial x}\Big(ka_x - lac_x\Big), \qquad c_t = Dc_{xx} + q_1 a - q_2 c.$$

(Details in lectures and in compendium)

Aggregation of Amoeba

Equilibrium points of (1):

= constant solutions of (1)

$\Rightarrow$ all constants (a_0, c_0) satisfying $\quad q_1 a_0 = q_2 c_0$.

Linear stability analysis:

Linearize equation around (a_0, c_0):

$a = a_0 + \tilde{a}, \quad c = c_0 + \tilde{c}; \quad \tilde{a}, \tilde{c}$ small; drop small(nonlin) terms

$$(2) \qquad \tilde{a}_t = k\tilde{a}_{xx} - la_0\tilde{c}_{xx}, \qquad \tilde{c}_t = D\tilde{c}_{xx} + q_1\tilde{a} - q_2\tilde{c}.$$

Stability: (a_0, c_0) is stable / unstable by linear stability analysis

if $\tilde{a}, \tilde{c}$ always remain small / do not remain small in all cases.

Aggregation of Amoeba

Particular solutions/Fourier modes of (2)**:** ($\approx$ heat equations)

$$(3) \qquad \tilde{a} = C_1 e^{\alpha t}\cos(\beta x) \qquad \text{and} \qquad \tilde{c} = C_2 e^{\alpha t}\cos(\beta x)$$

solve (2) iff

$$(4) \qquad \underbrace{\begin{pmatrix} \alpha + k\beta^2 & la_0\beta^2 \\ -q_1 & \alpha + D\beta^2 + q_2 \end{pmatrix}}_{B} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Analysis of (4)**:**

Non-zero solutions $\vec{C} = (C_1, C_2)$ iff $\det B = 0$

$\det B = 0 \quad \Leftrightarrow \quad \alpha^2 + b\alpha + c = 0$ for

$$(5) \qquad b = k\beta^2 + D\beta^2 + q_2 \quad \text{and} \quad c = kq_2\beta^2 + kD\beta^4 - q_1 la_0\beta^2.$$

Solve for α:

$\alpha_\pm = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$.

$b^2 - 4c = \cdots \geq 0 \quad \Rightarrow \quad \alpha_\pm \in \mathbb{R}, \quad (\alpha_- \leq \alpha_+)$.

Aggregation of Amoeba

Conclusions for (4):

(i) For very $\beta \in \mathbb{R}$, there are real $\alpha_\pm$ and solutions $\vec{C}_{\alpha_\pm,\beta} \neq 0$ of (4).

(ii) For any $s > 0$, $s\vec{C}_{\alpha_\pm,\beta}$ also solves (4).

(iii) Hence for every β and $\varepsilon > 0$,

there is a solution $\vec{C}_\beta$ of (4) with $\alpha = \alpha_+$ and $|\vec{C}_\beta| < \varepsilon$

Special solutions/Fourier modes (3):

(a) $(\tilde{a}, \tilde{c}) := \vec{C}_{\alpha_\pm,\beta} e^{\alpha_\pm t} \cos \beta t$ bounded $\Leftrightarrow$ $\alpha_+ \leq 0$ $\Leftrightarrow$ $c \geq 0$

(b) If $c < 0$, then $(\tilde{a}_1, \tilde{c}_1) := \vec{C}_\beta e^{\alpha_+ t} \cos \beta t$ unbounded, and

$$|\tilde{a}_1(0,x)|^2 + |\tilde{c}_1(x,0)|^2 = |\vec{C}_\beta|^2 < \varepsilon^2.$$

Aggregation of Amoeba

(Linearized) stability of (a_0, c_0):

Stable when $c \geq 0$ (all Fourier modes stable)

Unstable when $c < 0$ ($\tilde{a}_1, \tilde{c}_1$ starts near, but blows up)

In terms of the parameters of the problem:

$c \geq 0 \quad \Leftrightarrow \quad k(D\beta^2 + q_2) \geq q_1 l a_0,$ see (5)

Hence:

stable if $kq_2 \geq q_1 l a_0$ ($\Rightarrow$ $c \geq 0$ for all β)

unstable if $kq_2 < q_1 l a_0$ ($\Rightarrow$ $c < 0$ for β^2 small enough)

Aggregation of Amoeba – Conclusion

Parameters:

k diffusivity of amoeba a, q_2 break down rate of c,
q_1 production rate of c, la_0 attraction rate due to c.

Physical interpretation:

Enough food: $kq_2 \geq q_1 la_0$

$\Rightarrow$ (All) equilibrium point(s) (a_0, c_0) stable

$\Rightarrow$ solutions tend over time to constant / uniform consentration.

Lack of food: $kq_2 < q_1 la_0$

$\Rightarrow$ (All) equilibrium point(s) (a_0, c_0) unstable.

$\Rightarrow$ solutions move away from constant / uniform consentration.

This is the onset of aggregation/lumping!