

# The Method of Characteristics

Cauchy problem for quasi-linear PDE:

$$(1) \quad \begin{cases} u_t + a(t, x, u)u_x = b(t, x, u), & t > 0, \\ u(x, 0) = u_0(x), & t = 0. \end{cases}$$

Idea: Find  $(z(t), x(t))$  such that  $z(t) = u(x(t), t)$

$\Downarrow$  (1)

Characteristic equations:

$$(2) \quad \begin{cases} \dot{x} = a(x, y, z), & t > 0; & x(0) = x_0, \\ \dot{z} = b(x, y, z), & t > 0; & z(0) = u(x(0), 0) = u_0(x_0). \end{cases}$$

Implicit solution. Let  $X(t; x_0)$  and  $Z(t; x_0)$  be solutions of (2):

$$u(X(t; x_0), t) = Z(t; x_0)$$

Explicit solution. Set  $x = X(t; x_0)$ , invert,  $x_0 = X^{-1}(t, x)$ :

$$u(x, t) = Z(t; X^{-1}(t, x))$$