Convervation in Continuum Mechanics

The transport theorem: $R(t) = \{x(t) : \dot{x} = \vec{v}(x,t), x(t_0) = x_0 \in R(t_0)\}$

$$\boxed{\frac{d}{dt}\int_{R(t)} f(x,t)\,dx\Big|_{t=t_0} = \frac{d}{dt}\int_{R(t_0)} f(x,t)\,dx\Big|_{t=t_0} + \int_{\partial R(t_0)} f(x,t_0)\,(\vec{v}\cdot\vec{n})\,d\sigma}$$

Conservation of mass and momentum:

$$(1) \qquad \frac{d}{dt}\int_R \rho\,dx + \int_{\partial R} \rho\,(\vec{v}\cdot\vec{n})\,d\sigma = \int_R q\,dx.$$

$$(2) \qquad \frac{d}{dt}\int_R \rho\vec{v}\,dx + \int_{\partial R} \rho\vec{v}\,(\vec{v}\cdot\vec{n})\,d\sigma$$

$$\overset{\text{Transp.}}{\underset{\text{thm.}}{=}} \frac{d}{dt}\int_{R(t)} \rho\vec{v}\,dx \overset{\text{Newton's}}{\underset{\text{2nd law}}{=}} \underbrace{\int_{R(t)} \vec{f}_B\,dx}_{\text{body forces}} + \underbrace{\int_{\partial R(t)} \vec{f}_S\,d\sigma}_{\text{surface forces}}$$

Newtonian fluid: $\vec{f}_S = T\cdot\vec{n},\ T_{ij} = -(p + \frac{2}{3}\mu\nabla\cdot\vec{v})\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})$

Differential form - the Navier-Stokes equations:

$$(1') \quad \rho_t + \nabla\cdot(\rho\vec{v}) = q$$

$$(2') \quad (\rho v_i)_t + \nabla\cdot(\rho v_i\vec{v}) = f_{B,i} - \frac{\partial p}{\partial x_i} + \mu\big(\nabla^2 v_i + \frac{1}{3}\frac{\partial}{\partial x_i}(\nabla\cdot\vec{v})\big),\ i = 1,2,3$$