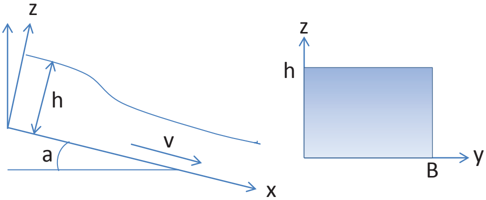
Flow in rivers – the shallow water equations



Assumptions:

1. v, h depend only on x, t; a small; $\rho = const.$
2. Dominant forces in x-direction:
   1. Gravity $\vec{f}_g \cdot \vec{e}_x = \rho g \sin a$
   2. Hydrostatic pressure $\vec{f}_p \cdot \vec{e}_x \approx -\rho g (h - z)(\vec{n} \cdot \vec{e}_x)$
   3. Bottom friction $\vec{f}_f \cdot \vec{e}_x = -\rho C_f v^2$

Control volume: $R = \{(x, y, z) : x \in [x_0, x_0 + \Delta x], y \in [0, B], z \in [0, h(x, t_0)]\}$

Conservation of mass and momentum in x-direction:

$$\frac{d}{dt} \int_R \rho \, dx + \int_{\partial R} \rho \, (\vec{v} \cdot \vec{n}) \, d\sigma = 0,$$

$$\frac{d}{dt} \int_R \rho v \, dx + \int_{\partial R} \rho v \, (\vec{v} \cdot \vec{n}) \, d\sigma = \int_R \vec{f}_g \cdot \vec{e}_x \, dx + \int_{\partial R} (\vec{f}_p + \vec{f}_f) \cdot \vec{e}_x \, d\sigma.$$

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Compute all integrals, divide by common factor ρB:

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} h\,dx + [(vh)(x_0+\Delta x) - (vh)(x_0)] = 0,$$

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} vh\,dx + [(v^2h)(x_0+\Delta x) - (v^2h)(x_0)]$$
$$= \int_{x_0}^{x_0+\Delta x} gh\sin a\,dx - \frac{g}{2}[h^2(x_0+\Delta x) - h^2(x_0)] - \int_{x_0}^{x_0+\Delta x} C_f v^2\,dx.$$

Divide by Δx, let $\Delta x \to 0$:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0,$$

$$\frac{\partial}{\partial t}(vh) + \frac{\partial}{\partial x}\left(v^2h + \frac{g}{2}h^2\right) = gh\sin a - C_f v^2.$$

This is the shallow water equations or St. Venant system.