

# Dimensional Analysis - Concepts

- Physical quantities:  $R_j = v(R_j)[R_j] = \text{value} \cdot \text{unit}$ ,  $j = 1, \dots, m$ .

Units:  $[R_j] = F_1^{a_{1j}} \cdots F_n^{a_{nj}}$ ,  $F_1, \dots, F_n$  fundamental units.

- Dimension matrix of  $R_1, \dots, R_m$ :

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \quad (n \times m)$$

- Dimensionless combination:

$$\pi = R_1^{\lambda_1} \cdots R_m^{\lambda_m} \quad \text{where} \quad \vec{\lambda} \neq 0 \quad \text{and} \quad [\pi] = 1$$

- Dimensionally independent  $R_1, \dots, R_s$

- if no dimensionless combinations exists

# Dimensional Analysis - Change of units

- **Change of units**  $\Rightarrow$  change of values of  $R_j$ :

$$F_i = x_i \hat{F}_i, x_i > 0 \quad \Rightarrow \quad \hat{v}(R_j) = x_1^{a_{1j}} \dots x_n^{a_{nj}} v(R_j)$$

- **Physical relation**  $\Phi(R_1, \dots, R_m) = 0$  is dimensionally consistent if

$$\Phi(v(R_1), \dots, v(R_m)) = 0$$



$$\Phi(\hat{v}(R_1), \dots, \hat{v}(R_m)) = 0$$

for all changes of units  $\hat{F}_i$ . (Consistent under change of units).

# Dimensional Analysis - Buckingham's pi-theorem

**Assume:**

- (A1)  $F_1, \dots, F_n$  are fundamental units
- (A2)  $R_1, \dots, R_m$  are physical quantities
- (A3)  $\Phi(R_1, \dots, R_m) = 0$  is dimensionally consistent

**Define:**  $r = \text{rank } A$  (= number of linearly independent columns in  $A$ )  
where  $A$  is  $n \times m$  dimension matrix of  $R_1, \dots, R_m$ .

**Buckingham's pi-theorem:**

- (a) There are exactly  $m - r$  independent dimensionless combinations.
- (b) For any set of  $m - r$  independent dimensionless combinations  $\pi_1, \dots, \pi_{m-r}$ , there is a relation  $\Psi$  such that

$$\Phi(R_1, \dots, R_m) = 0 \quad \Leftrightarrow \quad \Psi(\pi_1, \dots, \pi_{m-r}) = 0.$$

It remains to prove the **pi-theorem**.

# Example: From first atomic explosion

Assumption (G.I. Taylor): There exists  $\Phi$  s.t.  $\Phi(r, t, E, \rho) = 0$

1 Dimension matrix:

$$A = \begin{array}{c|cccc} & r & t & \rho & E \\ \hline m & 1 & 0 & -3 & 2 \\ s & 0 & 1 & 0 & -2 \\ kg & 0 & 0 & 1 & 1 \end{array}$$

2 Dimensionless combinations:

$$r := \text{rank } A = 3 \quad \Rightarrow \quad m - r = 4 - 3 = 1 \text{ dim.less combination}$$

$$r, t, g \text{ dim. independent} \quad \Rightarrow \quad \pi = \frac{E}{r^\bullet t^\bullet \rho^\bullet} = \dots = \frac{E}{r^5 t^{-2} \rho}$$

3 Buckingham's pi-theorem:

$$\Phi(\dots) = 0 \quad \Leftrightarrow \quad \Psi(\pi) = 0$$