Scaling and non-dimensionalizing

Produce dimensionless variables and equations, with variables and coefficients $\lesssim 1$.

Scaling a variable u^*: $\boxed{u^* = Uu}$

scaled variable: $u \sim 1$, $[u] = 1$

scaling constant: $U \sim \max |u^*|$, $[U] = [u^*]$

Scaling/nondimensionalizing an equation:

1. scaling all variables in the equation
2. dividing the resulting equation by $\sim$ biggest coefficient.

Finding scales

Scales are combinations of the parameters of the problem

Recipe:

1. Introduce scaled variables into all equations and constraints (scales to be determined): $x^* = Xx, t^* = Tt, \frac{dx^*}{dt^*} = \frac{X}{T}\frac{dx}{dt}$...
2. Scaling assumption: Scales variables and derivatives ~ 1: $t, x(t), x'(t), \dots \sim 1$.
3. Determine scales $X, T, \dots$ by either of:
   a) determining directly $\max|t^*|$, $\max|x^*|$, ...
   b) balance 2 dominating/biggest terms in equation
   c) solve a reduced problem to find estimates

Remarks on scaling

When 2 dominating terms balances:

$\Rightarrow$ their coefficients are equal and $\sim$ biggest in equation.

$\Rightarrow$ dividing scaled equation by this coefficient:
All variables and coefficients become

- dimensionless,
- $\lesssim 1$, and
- 2 coefficients $= 1$.

Different situations $\Rightarrow$ different scales:

$\max|x^*|$, $\max|t^*|$, $\max|u^*|$, etc., and hence the dominating terms in the equation depend on the situation.

Remarks on scaling

Typical time scale for $u^*(t^*)$:

$$T = \frac{\max |u^*|}{\max \left|\frac{du^*}{dt^*}\right|}$$

Advantages of scaling:

- minimize the number of parameters/coefficients,
- normalize all variables and coefficients,
- reduce round-off errors in numerical calculations,
- makes small terms visible
  $\Rightarrow$ easy to do approximations/perturbation.