Singular Perturbation

Signs:

- Multiple time/space scales
- Initial/boundary layers
- Small paramater multiplying principal term
- Naive approximation changes problem completely

Facts:

- No single scale is good for complete solution
- Different regions, different scales, different (re)scaled eq'ns
- In each region regular perturbation works
- Balance terms in equation to find scales
- Matching of solutions from different regions

Singular Perturbation – 1st approximation

$$\varepsilon y'' + 2y' + y = 0, \ 0 < x < 1; \quad y(0) = 0, \ y(1) = 1; \quad 0 < \varepsilon \ll 1.$$

- 1. Guess where boundary layer is: x = a. Here a = 0.
- 2. Outer solution y_O . Set $\varepsilon = 0$, solve equation and boundary condition outside boundary layer:

$$2y'_O + y_O = 0$$
; $y_O(1) = 1 \implies y_O(x) = e^{\frac{1}{2} - \frac{x}{2}}$.

3. Balance terms to find length of boundary layer δ , the other *consistent* space scale: ... $\delta = \varepsilon$.

Singular Perturbation – 1st approximation

- 3. Find length of boundary layer $\delta \dots \delta = \varepsilon$.
- 4. Rescale equation: $(x, y) = (\delta \xi, Y)$

$$\Rightarrow Y''(\xi) + 2Y'(\xi) + \varepsilon Y(\xi) = 0$$

5. Inner solution y_l . Set $\varepsilon = 0$ and solve rescaled equation and boundary condition inside boundary layer.

$$y_l'' + 2y_l' = 0$$
, $y_l(0) = 0$ \Rightarrow $y_l(\xi) = C(1 - e^{-2\xi})$.

- 6. Matching. $y_0 \approx y_1$ in intermediate region $\ldots C = e^{\frac{1}{2}}$
- 7. Uniform solution: $y_U(x) = y_O(x) + y_I(\frac{x}{\delta}) \lim_{x \to 0} y_O(x)$