The Method of Caracteristics

Cauchy problem for quasi-linear PDEs:

$$(1) \qquad \begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) & \text{in } \Omega \subset \mathbb{R}^2, \\ u(x,y) = \bar{h}(x,y) & \text{on curve } \gamma : (f(s), g(s)). \end{cases}$$

Idea: PDE $\rightarrow$ ODEs, $u(x,y) \rightarrow (x(t), y(t), z(t))$ where $z(t) = u(x(t), y(t))$.

Characteristic equations:

$$(2) \qquad \begin{cases} \dot{x} = a(x,y,z), & t > 0; & x(0) = f(s), \\ \dot{y} = b(x,y,z), & t > 0; & y(0) = g(s), \\ \dot{z} = c(x,y,z), & t > 0; & z(0) = \bar{h}(f(s), g(s)). \end{cases}$$

Implicit solution: $u(X(t,x), Y(t,s)) = Z(t,s)$ when (X, Y, Z) solve (2).

Explicit solution: $u(x,y) = Z(T(x,y), S(x,y))$ when $\binom{x}{y} = \binom{X(t,s)}{Y(t,s)} \underset{invert}{\longrightarrow} \binom{t}{s} = \binom{T(x,y)}{S(x,y)}$

Theorem:
The method works and produce the unique C^1 solution $u(x,y) = Z(T(x,y), S(x,y))$ of (1) close to $(x_0, y_0) \in \gamma$ if

(i) $a, b, c, \bar{h}$ is C^1 near $P_0 = (x_0, y_0, \bar{h}(x_0, y_0))$

(ii) γ is C^1 and non-characteristic: γ not parallel to (a, b) at P_0, *or equivalently*

$$\begin{vmatrix} f'(s_0) & g'(s_0) \\ a(P_0) & b(P_0) \end{vmatrix} \neq 0 \quad \text{where} \quad f(s_0) = x_0.$$