Notes on conservation laws and continuum mechanics

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The divergence theorem revisited

I shall follow the textbook in using only a single integral sign even for multiple integrals in the abstract. (If I ever need the iterated integrals, I shall of course write out all the integral signs.)

The divergence theorem is then written as

$$\int_R \nabla\cdot\mathbf{F}\,\mathrm{d}V = \int_{\partial V} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

in three dimensions.

It works just as well in two dimensions, if we replace the volume element $\mathrm{d}V$ by the area element $\mathrm{d}A$ and the surface element $\mathrm{d}S$ by $\mathrm{d}s$ where s is arclength: Then it is simply Green's theorem

$$\int_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathrm{d}A = \int_{\partial R} P\,\mathrm{d}x + Q\,\mathrm{d}y$$

if we note that the outward pointing unit normal is $\mathbf{n} = (\mathrm{d}y/\mathrm{d}s, -\mathrm{d}x/\mathrm{d}s)$ and we put $\mathbf{F} = (Q, -P)$.

It even works in one dimension: A vector field in one dimension is nothing but a function f, its divergence is the derivative f', the boundary of an interval $R = [a, b]$ is the set $\{a, b\}$ of end points, and then it is just the fundamental theorem of calculus

$$\int_{[a,b]} f'(x)\,\mathrm{d}x = f(b) - f(a)$$

where the righthand side is suitably considered as a zero-dimensional integral, i.e., a sum of fn over the set $\{a, b\}$ where the outside normal n is $+1$ at b and -1 at a.

In the following, I shall mostly stick to three dimensions. I hope it is clear from the above how the results can be applied in one or two dimensions as well.

Flux and conservation

A *flux* is defined to be the amount of some quantity moving across a given surface per unit time.[1] Very often, a flux can be derived from a *flux density*, which is a vector field $\boldsymbol{\phi}$ so that the corresponding flux across a surface S is

$$\Phi = \int_S \boldsymbol{\phi}\cdot\mathbf{n}\,\mathrm{d}S.$$

A *conservation law* is an equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R \rho\,\mathrm{d}V + \int_{\partial R} \boldsymbol{\phi}\cdot\mathbf{n}\,\mathrm{d}S = \int_R q\,\mathrm{d}V$$

with the interpretation that ρ is a *density* of something, so that the $\int_R \rho\,\mathrm{d}V$ is the amount of this something inside R, the surface integral is the flux across the boundary of R, and $\int_R q\,\mathrm{d}V$ is a *source term*.

For example, ρ could be ordinary mass density, in which case the source term is usually zero. Or ρ could be a concentration of some chemical, in which case q would represent the result of a chemical reaction involving the chemical (creating or destroying it). In the latter case, the flux density might arise from a combination of movement of a macroscopic medium (fluid?) and molecular diffusion of the chemical within the fluid.

Provided that the functions involved are sufficiently smooth (C^1 is usually enough), the general conservation law can be rewritten as

$$\int_R \frac{\partial\rho}{\partial t}\,\mathrm{d}V + \int_R \nabla\cdot\boldsymbol{\phi}\,\mathrm{d}V = \int_R q\,\mathrm{d}V.$$

If this is assumed to hold for *every* region R, we can apply the Raymond–Dubois theorem[2] to conclude that

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{\phi} = q.$$

For example, in a continuum with mass density ρ and velocity v, the mass flux is ρv and $q = 0$, so we get the law of mass conservation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{1}$$

[1] This ignores quantities like magnetic flux, which is of course interesting but not our concern presently.

[2] If f is locally integrable and $\int_R f\,dV = 0$ for every region R, then $f = 0$ *almost everywhere*. If you don't know what that means, don't worry. For a continuous function f, it is equivalent to $f = 0$ everywhere. The Raymond–Dubois theorem for continuous functions is *really* trivial.

The transport theorem and its applications

Some times, conservation laws are more easily expressed in terms of a moving volume. The transport theorem[3] allows us to transform such a conservation law to one that involves a fixed volume instead.

The notation tends to get hairy, so let us fix our attention on a specific time $t = 0$. A region $R(t)$ is assumed to move with the velocity $\mathbf{v}(\mathbf{x}, t)$, which simply means that, whenever a point with coordinates $\mathbf{x}(t)$ moves with the velocity field, i.e., it satisfies the differential equation $\dot{\mathbf{x}} = \mathbf{v}(x, t)$, then either $\mathbf{x}(t) \in R(t)$ for all t, or $\mathbf{x}(t) \notin R(t)$ for all t.

In this case, the transport theorem states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R(t)} \rho\,\mathrm{d}V\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{R(0)} \rho\,\mathrm{d}V\bigg|_{t=0} + \int_{\partial R(0)} \rho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S.$$

The first term on the righthand side is interpreted as the change due to changes in the field ρ, wherease the second term comes from the movement of the boundary with velocity v. Intuitively, where $\mathbf{v}\cdot\mathbf{n} > 0$, the boundary is moving outward, extending the region R, and so including more of ρ.

There is of course nothing special about the time $t = 0$. We often write the transport theorem in the simplified form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R(t)} \rho\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{R} \rho\,\mathrm{d}V + \int_{\partial R} \rho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S.$$

where it is to be understood that the region R is supposed to be kept fixed while we differentiate the first integral on the righthand side: But then we put $R = R(t)$ afterwards.

One application of the transport theorem is to the *conservation of momentum* for a continuum, which we write on the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R(t)} \rho\mathbf{v}\,\mathrm{d}V = \int_{\partial R} \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}S + \int_{R} \rho\mathbf{g}\,\mathrm{d}V.$$

Don't worry too much about the mysterious term $\mathbf{n}\cdot\boldsymbol{\sigma}$ yet: It is merely the force per unit area, or surface stress, acting on the surface of R. The final term is due to gravity, but could be generalized in more or less obvious ways to other body forces such as electromagnetic forces, for example. My point here is to point out that we can get rid of the troublesome moving region $R(t)$ by employing the transport theorem to rewrite the lefthand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R} \rho\mathbf{v}\,\mathrm{d}V + \int_{\partial R} (\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}S = \int_{\partial R} \mathbf{n}\cdot\boldsymbol{\sigma}\,\mathrm{d}S + \int_{R} \rho\mathbf{g}\,\mathrm{d}V. \qquad (2)$$

[3] I am not too sure how widespread this name is.

The new lefthand side can be interpreted as the rate of change of momentum inside the fixed region R, plus the flux of momentum carried with the continuum across the boundary of R.

It would seem very questionable to write up this equation from scratch, since we relied on the usual notation of a physical system as consisting of a fixed collection of matter.

We can apply the same thinking to *energy conservation*: The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R} e\,\mathrm{d}V + \int_{\partial R} e\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\partial R} \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{v}\,\mathrm{d}S$$

where e is the energy density, and the righthand side is the work done by the surface forces on the system. But haven't I forgotten the work of gravity? No, I have incorporated that into e by writing $e = u + \frac{1}{2}\rho v^2 + \rho g z$, where u is the internal energy.

To continue the example, assume the continuum under study is an inviscid fluid. Then $\mathbf{n}\cdot\boldsymbol{\sigma} = -p\mathbf{n}$, where p is the pressure. Assume further that we're looking at a steady state. Then the first time derivative is zero, and so we're left with[4]

$$\int_{\partial R} (e + p)\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = 0.$$

Now, let R be a stream tube: I.e., a long thin tube whose sides consists of stream lines of the flow. Thus $\mathbf{v}\cdot\mathbf{n} = 0$ along those sides, and so only the two ends of the stream tube contribute to the integral. At the ends, we may assume that e and p are constants (different constants at the two ends of course), and $\mathbf{v}\cdot\mathbf{n} = \pm v$ is simularly constant at each end. Finally, the total mass flux $\rho v \Delta S$ is constant along the tube, and so we are led to the conclusion that $(e - p)/\rho$ is the same at each end; i.e., this quantity is constant along stream lines.

This is nothing but *Bernoulli's law*: If the fluid is incompressible and no heat transfer occurs, we may ignore the internal energy u, and so write $(e - p)/\rho = \frac{1}{2}v^2 + gz + p/\rho$ for the quantity that is constant along stream lines.

The stress tensor

The surface stress across a surface S in a continuum is a vector field $\mathbf{T}$ on the surface so that the total force across the surface is

$$\int_{S} \mathbf{T}\cdot\mathbf{n}\,\mathrm{d}S.$$

[4] You should convince yourself that e, the energy density (energy per volume) and p, the pressure (force per unit area) have the same dimensions.

To be precise, this is the force *by* the side which $\mathbf{n}$ points into, acting *on* the side which $\mathbf{n}$ points out of.

Clearly, T is a function of position, time and the normal vector, so we should write it $\mathbf{T}(\mathbf{x}, t, \mathbf{n})$.

Newton's third law immediately tells us that $\mathbf{T}(\mathbf{x}, t, -\mathbf{n}) = -\mathbf{T}(\mathbf{x}, t, \mathbf{n})$.

Moreover, it can be shown under quite moderate assumptions that $\mathbf{T}$ must be a linear function of $\mathbf{n}$. Now that makes little sense perhaps, since $\mathbf{n}$ is constrained to be a unit vector. But we help things along by extending the definition of $\mathbf{T}$ by setting $\mathbf{T}(\mathbf{x}, t, \alpha\mathbf{n}) = \alpha\mathbf{T}(\mathbf{x}, t, \mathbf{n})$ for any $\alpha > 0$. Since we have just seen that it also holds for $\alpha = -1$, it must then hold for all real α, so only the additivity remains to prove.

We can show this by considering a small prism with parallell end surfaces and a triangular cross-section. If $\mathbf{T}$ is not additive, it turns out the prism must accelerate with a speed that goes to infinity as the size of the prism goes to zero.[5]

This leads then to the definition of the *stress tensor*: Remember from linear algebra that all linear maps can be implemented by matrix multiplication. The stress tensor $\boldsymbol{\sigma}$ is the required 3×3 matrix, so that

$$\mathbf{T}(\mathbf{x}, t, \mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\sigma}.$$

A consideration of angular momentum and small regions leads to the conclusion that $\boldsymbol{\sigma}$ must be *symmetric*, or else small region will achieve an ubounded angular momentum.

Continuum mechanics

We can now get back to (2) and apply the standard tricks of the trade to it, with the result

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \boldsymbol{\sigma} + \rho\mathbf{g}.$$

Here, the tensor product $\rho\mathbf{v} \otimes \mathbf{v}$ is defined as the tensor with (i, j) component $\rho v_i v_j$, and $\nabla \cdot \boldsymbol{\sigma}$ is the vector with j component

$$\sum_i \frac{\partial \sigma_{ij}}{\partial x_i}$$

and similarly for $\nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v})$.[6] If we use the product rule can write this as

$$\frac{\partial \rho}{\partial t}\mathbf{v} + \rho\frac{\partial \mathbf{v}}{\partial t} + \big(\nabla \cdot (\rho\mathbf{v})\big)\mathbf{v} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \cdot \boldsymbol{\sigma} + \rho\mathbf{g}.$$

[5] I would write out the details, but time does not permit this right now.

[6] You should convince yourself that the divergence theorem works for tensor fields, by writing out the usual divergence theorem in component form.

The thing to notice here is that the first and third terms cancel because of mass conservation (1), and so we are left with

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \frac{\nabla \cdot \boldsymbol{\sigma}}{\rho} + \mathbf{g} \tag{3}$$

after dividing by ρ. This is the *general equation of motion of a continuum*, so long as no discontinuities appear. (Except we may still wish to generalize the body force term $\mathbf{g}$.)

Newtonian fluids

An incompressible *Newtonian fluid* is one in which we can write the (i, j) component of the stress tensor as

$$\sigma_{ij} = -p\delta_{ij} + \mu\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)$$

where δ is ths usual Kronecker delta: $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$. (Matters get a bit more complicated for compressible fluids.) Plugging that into (3) we quickly arrive at

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g},$$

which is the *Navier–Stokes equation* for an incompressible fluid.

Above, μ is the *dynamic viscosity* of the fluid, and $\nu = \mu/\rho$ is the *kinematic viscosity* of the fluid.

For an inviscid fluid, i.e., one with zero viscosity, we are then left with the *Euler equation*

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g},$$

which has a much wider applicability than you'd think, because it can often be used away from a turbulent boundary layer. The basic theory of airfoils, for example, is largely based on the Euler equations. This is perhaps more easily seen from the scaled Navier–Stokes equation (without gravity):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{v},$$

which becomes the Euler equation if we let $\mathrm{Re} \to \infty$. (But this is a *singular perturbation*: It turns a second order equation into a first order one.)