Dimensional Analysis - Concepts

- **Physical quantities:** $R_j = v(R_j)[R_j] =$ value $\cdot$ unit, $\quad j = 1, \ldots, m$.

  Units: $\quad [R_j] = F_1^{a_{1j}} \cdots F_n^{a_{nj}}, \quad F_1, \ldots, F_n$ fundamental units.

- **Dimension matrix** of $R_1, \cdots, R_m$:

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nm} \end{bmatrix} \qquad (n \times m)$$

- **Dimensionless combination:**

$$\pi = R_1^{\lambda_1} \cdots R_m^{\lambda_m} \qquad \text{where} \qquad \vec{\lambda} \neq 0 \quad \text{and} \quad [\pi] = 1$$

- **Dimensionally independent** $R_1, \ldots, R_s$

  - if no dimensionless combinations exists

Dimensional Analysis - Change of units

- **Change of units** $\Rightarrow$ change of values of R_j:

$$F_i = x_i \hat{F}_i,\; x_i > 0 \quad \Rightarrow \quad \hat{v}(R_j) = x_1^{a_{1j}} \dots x_n^{a_{nj}} v(R_j)$$

- **Physical relation** $\Phi(R_1, \dots, R_m) = 0$ is dimensionally consistent if

$$\Phi(v(R_1), \dots, v(R_m)) = 0$$
$$\Updownarrow$$
$$\Phi(\hat{v}(R_1), \dots, \hat{v}(R_m)) = 0$$

for all changes of units $\hat{F}_i$. (Consistent under change of units).

Dimensional Analysis - Buckingham's pi-theorem

Assume:

(A1) $F_1, \ldots, F_n$ are fundamental units

(A2) $R_1, \ldots, R_m$ are physical quantities

(A3) $\Phi(R_1, \ldots, R_m) = 0$ is dimensionally consistent

Define: $r = \operatorname{rank} A$ (= number of linearly independent collumns in A) where A is $n \times m$ dimension matrix of $R_1, \ldots, R_m$.

Buckingham's pi-theorem:

(a) There are exactly $m - r$ independent dimensionless combinations.

(b) For any set of $m - r$ independent dimensionless combinations $\pi_1, \ldots, \pi_{m-r}$, there is a relation Ψ such that

$$\Phi(R_1, \ldots, R_m) = 0 \quad \Leftrightarrow \quad \Psi(\pi_1, \ldots, \pi_{m-r}) = 0.$$

It remains to prove the pi-theorem.

Example: From first atomic explosion

Assumption (G.I. Taylor): There exists Φ s.t. $\Phi(r, t, E, \rho) = 0$

1. Dimension matrix:

$A =$

	r	t	ρ	E
m	1	0	-3	2
s	0	1	0	-2
kg	0	0	1	1

2. Dimensionless combinations:

$r := \mathrm{rank}\, A = 3 \quad \Rightarrow \quad m - r = 4 - 3 = 1$ dim.less combination

r, t, g dim. independent $\quad \Rightarrow \quad \pi = \frac{E}{r^{\bullet} t^{\bullet} \rho^{\bullet}} = \cdots = \frac{E}{r^5 t^{-2} \rho}$

3. Buckingham's pi-theorem:

$$\Phi(\dots) = 0 \quad \Leftrightarrow \quad \Psi(\pi) = 0$$