# Singular Perturbation

Signs:

- Multiple time/space scales
- Initial/boundary layers
- Small paramater multiplying principal term
- Naive approximation changes problem completely

Facts:

- No single scale is good for complete solution
- Different regions, different scales, different (re)scaled eq'ns
- In each region regular perturbation works
- Balance terms in equation to find scales
- Matching of solutions from different regions

# Singular Perturbation – 1st approximation

$$\varepsilon y'' + 2y' + y = 0,\ 0 < x < 1; \quad y(0) = 0,\ y(1) = 1; \quad 0 < \varepsilon \ll 1.$$

1. Guess where boundary layer is: $x = a$. Here $a = 0$.
2. Outer solution $y_O$. Set $\varepsilon = 0$, solve equation and boundary condition outside boundary layer:

$$2y_O' + y_O = 0;\ y_O(1) = 1 \implies y_O(x) = e^{\frac{1}{2} - \frac{x}{2}}.$$

3. Balance terms to find length of boundary layer $\delta$, the other *consistent* space scale: $\ldots\ \delta = \varepsilon$.

# Singular Perturbation – 1st approximation

3. Find length of boundary layer $\delta$ ... $\delta = \varepsilon$.

4. Rescale equation: $(x, y) = (\delta\xi, Y)$

$$\Rightarrow\ Y''(\xi) + 2Y'(\xi) + \varepsilon Y(\xi) = 0$$

5. Inner solution $y_I$. Set $\varepsilon = 0$ and solve rescaled equation and boundary condition inside boundary layer.

$$y_I'' + 2y_I' = 0,\ y_I(0) = 0 \quad \Rightarrow \quad y_I(\xi) = C(1 - e^{-2\xi}).$$

6. Matching. $y_O \approx y_I$ in intermediate region $\underset{\varepsilon \to 0}{\dots}$ $\quad C = e^{\frac{1}{2}}$

7. Uniform solution: $y_U(x) = y_O(x) + y_I(\frac{x}{\delta}) - \lim_{x \to 0} y_O(x)$