The Method of Caracteristics

Cauchy problem for quasi-linear PDE:

$$(1) \qquad \begin{cases} u_t + a(t,x,u)u_x = b(t,x,u), & t > 0, \\ u(x,0) = u_0(x), & t = 0. \end{cases}$$

Idea: Find $(z(t), x(t))$ such that $\boxed{z(t) = u(x(t), t)}$

$$\Downarrow (1)$$

Characteristic equations:

$$(2) \qquad \begin{cases} \dot{x} = a(x,y,z), & t > 0; \quad x(0) = x_0, \\ \dot{z} = b(x,y,z), & t > 0; \quad z(0) = u(x(0),0) = u_0(x_0). \end{cases}$$

Implicit solution. Let $X(t; x_0)$ and $Z(t; x_0)$ be solutions of (2):

$$\boxed{u(X(t;x_0),t) = Z(t;x_0)}$$

Explicit solution. Set $x = X(t; x_0)$, invert, $x_0 = X^{-1}(t,x)$:

$$\boxed{u(x,t) = Z(t; X^{-1}(t,x))}$$