

# Conservation in Continuum Mechanics

**The transport theorem:**  $R(t) = \{x(t) : \dot{x} = \vec{v}(x, t), x(t_0) = x_0 \in R(t_0)\}$

$$\frac{d}{dt} \int_{R(t)} f(x, t) dx \Big|_{t=t_0} = \frac{d}{dt} \int_{R(t_0)} f(x, t) dx \Big|_{t=t_0} + \int_{\partial R(t_0)} f(x, t_0) (\vec{v} \cdot \vec{n}) d\sigma$$

**Conservation of mass and momentum:**

$$(1) \quad \frac{d}{dt} \int_R \rho dx + \int_{\partial R} \rho (\vec{v} \cdot \vec{n}) d\sigma = \int_R q dx.$$

$$(2) \quad \frac{d}{dt} \int_R \rho \vec{v} dx + \int_{\partial R} \rho \vec{v} (\vec{v} \cdot \vec{n}) d\sigma$$

$$\begin{array}{ccc} \text{Transp.} & & \text{Newton's} \\ \text{thm.} & \frac{d}{dt} \int_{R(t)} \rho \vec{v} dx & \text{2nd law} \int_{R(t)} \vec{f}_B dx + \int_{\partial R(t)} \vec{f}_S d\sigma \\ & & \text{body forces} \quad \text{surface forces} \end{array}$$

**Newtonian fluid:**  $\vec{f}_S = T \cdot \vec{n}$ ,  $T_{ij} = -(\rho + \frac{2}{3}\mu \nabla \cdot \vec{v})\delta_{ij} + \mu(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})$

**Differential form - the Navier-Stokes equations:**

$$(1') \quad \rho_t + \nabla \cdot (\rho \vec{v}) = q$$

$$(2') \quad (\rho v_i)_t + \nabla \cdot (\rho v_i \vec{v}) = f_{B,i} - \frac{\partial p}{\partial x_i} + \mu(\nabla^2 v_i + \frac{1}{3} \frac{\partial}{\partial x_i} (\nabla \cdot \vec{v})), \quad i = 1, 2, 3$$